ficients in Eq. (23). In addition, they are used to find a general expression for the coefficient of the leading term of $G^j(s)$. We calculate the determinant $\mathrm{Det}[e(p_i,q_k)]$ for the case in which each set consists of the numbers 0, 1, 2, $\cdots$, $(j-1)$. If the rows and columns are arranged so that all the odd indices appear first, the determinant is in clearly factorable form, with zeros in all positions of the two off-diagonal (even-odd, odd-even) blocks. The dimensions of the factors will be equal or will differ by one, depending on whether j is even or odd. The p_iq_k element of either factor is $(p_i+q_k+1)^{-1}$. Either diagonal block is designated as $D(m)$, where m is the largest value of p_i or q_k. Evaluation of $D(m)$ is straightforward and may be found in the treatise by Muir and Metzler[16]:

$$D(2u+\alpha)=\prod_{t=0}^{u}\left[\frac{(2t+\alpha)!}{(4t+2\alpha+1)!!}\right]^2 \times(4t+2\alpha+1)\,;\quad \alpha=0 \text{ or } 1. \qquad \text{(A5)}$$

Whether j is odd or even, one may write $\mathrm{Det}[e(p_i,q_k)]$ as $D(j)D(j-1)$. Evaluating this product from Eq. (A5) and substituting the result in Eq. (23) yields, for the leading term of $G^j(s)$

$$\left(\frac{2}{\pi}\right)^j\left(\frac{\pi s}{2}\right)^{j^2}\left[\prod_{k=1}^{j}\frac{k!}{(2k+1)!!(2k-1)!!}\right]^2. \qquad \text{(A6)}$$

[16] T. Muir and W. H. Metzler, *A Treatise on the Theory of Determinants* (Dover Publications, Inc., New York, 1960), p. 429.

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Gravitational Field: Equivalence of Feynman Quantization and Canonical Quantization

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The transition amplitude for the gravitational field as given by the Feynman sum over histories expression is analyzed in analogy to the electromagnetic transition amplitude. The analysis is based on an explicit representation of the Feynman sum by means of a lattice. The measure is found by consistency requirements and differs from those proposed by other workers. Particular attention is paid to the subsidiary conditions associated with the gauge group. It is shown, that the present approach is equivalent to the quantization by means of canonical variables as proposed by Dirac.

I. INTRODUCTION

THIS paper deals with the problem of assigning a well-defined meaning to

$$\sum_{\text{histories}} e^{iS}, \qquad \text{(I.1)}$$

if S is the action for the free gravitational field. The present approach may actually be extended to the more general case of gravity interacting with matter. For simplicity we shall deal with the gravitational field only.

The prescription given by Feynman[1] to compute (I.1) is not completely straightforward, because the action for the gravitational field is degenerate. The presence of an invariance group generates various difficulties which are well known for the case of the electromagnetic field and its Abelian gauge group. The quantization of the electromagnetic field in the framework of the Feynman sum over histories is analyzed in some detail in Sec. II and constitutes the basis of the present approach to the quantization of the free gravitational field. In particular, we examine the subsidiary condition associated with the gauge group, which in the case of the electromagnetic transition amplitude states that this amplitude is invariant with respect to a gauge transformation of the potential at the initial and the final surface. Section III deals with the generalization of this discussion to the gravitational case in a purely formal and heuristic manner. A more precise framework for the evaluation of the gravitational amplitude is set up in Sec. IV and the derivation of the subsidiary conditions in this framework is given in Sec. V where we also proceed to convert them into differential form. Finally, it is shown in Sec. VI that the results obtained are equivalent to the results of the Hamiltonian quantization procedure as proposed by Dirac.[2] One could and should trace out in a similar way the connection between the sum over histories formulation and the canonical formalism given by Arnowitt, Deser, and Misner.[2] However, to treat this connection would lengthen the present account unduly.

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[1] R. P. Feynman, Rev. Mod. Phys. **20**, 267 (1948).

[2] P. A. M. Dirac, Proc. Roy. Soc. (London) **A246**, 333 (1958); Phys. Rev. **114**, 924 (1959); R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **113**, 745 (1959); **116**, 1322 (1959); **117**, 1595 (1960); **118**, 1100 (1960).

Definition of the Sum Over Histories on a Lattice of Points

We adopt the original definition of the Feynman sum over histories, where the integrals involved are defined on a lattice of points. In our case, the lattice covers a region in four-dimensional space time between two space-like hypersurfaces τ' and τ''. The lattice will then consist of a family of hypersurfaces between τ' and τ'' and a family of curves connecting τ' and τ'', whose intersections with the surfaces are the points of the lattice. The transition amplitude from one member of the family of hypersurfaces to the next one is set proportional to[3] $\exp iS^0$ where S^0 is the classical value of the action S for stationary histories. The action is specified as

$$S=\int \mathfrak{L}(g)d^4x\,;\qquad \mathfrak{L}(g)=|g|^{1/2}g^{\mu\nu}(\Gamma_{\mu\nu}{}^{\rho}\Gamma_{\rho\sigma}{}^{\sigma}-\Gamma_{\mu\rho}{}^{\sigma}\Gamma_{\sigma\nu}{}^{\rho}). \tag{I.2}$$

We write the amplitude formally as

$$\mathfrak{N}\int e^{iS}\mathfrak{D}g\,, \tag{I.3}$$

where the measure $\mathfrak{D}g$ includes the product of the differentials $dg_{00}\cdots dg_{33}$ for each point of the lattice. The action S is not a quadratic functional of its arguments. We therefore have to expect that the measure will depend on the history $g_{\mu\nu}(x)$.

Analogy: Free Particle in Spherical Coordinates

In order to understand how this comes about, let us briefly investigate the transition amplitude for a nonrelativistic free particle in spherical coordinates. Here the action is

$$S=\frac{m}{2}\int\{\dot r^2+r^2(\dot\theta^2+\dot\varphi^2\sin^2\theta)\}dt. \tag{I.4}$$

What is the analog of $\mathfrak{D}g$ in this case?

According to Feynman's definition, the infinitesimal amplitude is given by[4]

$$(q'',t'+\epsilon|q',t')=N_\epsilon e^{iS^0(q''t'+\epsilon|q't')}(1+o(\epsilon))\,, \tag{I.5}$$

where q is shorthand for r, θ, φ, and $o(\epsilon)$ indicates that terms of higher order than ϵ are irrelevant; in what follows limit $\epsilon\to 0$ will always be understood and we will disregard the terms vanishing in the limit. The amplitude for a finite time inverval is defined by

$$(q'',t''|q',t')=\int\prod_{i=0}^{N}(q_{i+1},t_{i+1}|q_i,t_i)\mathfrak{D}q\,, \tag{I.6}$$

where

$$t''-t'=\epsilon(N+1)\,,\quad t_i=t+i\epsilon\,,\quad q_{N+1}=q''\,,\quad q_0=q'.$$

We write the measure $\mathfrak{D}q$ as

$$\mathfrak{D}q=\prod_{i=1}^{N}f(q_i)dr_id\theta_id\varphi_i\,, \tag{I.7}$$

with an as yet unknown function $f(q)$. Equations (I.5), (I.6), and (I.7) define the amplitude and give meaning to the formal expression

$$(q'',t''|q',t')=\mathfrak{N}\int e^{iS}\mathfrak{D}q. \tag{I.8}$$

What we want to emphasize is that the function f which appears in the measure $\mathfrak{D}q$ is determined by consistency. For a given action there is only one measure such that the limit $\epsilon\to 0$ exists. To show how $f(q)$ is determined by the action, consider Eq. (I.6) and insert explicitly the infinitesimal amplitude from t_N to t''.

$$(q'',t''|q',t')=\int N_\epsilon e^{iS^0(q'',t''|q,t''-\epsilon)}\times f(q)drd\theta d\varphi(q,t''-\epsilon|q',t'). \tag{I.9}$$

As $\epsilon\to 0$, $N\to\infty$ the amplitude from t' to $t''-\epsilon$ converges to the amplitude from t' to t''; therefore, we conclude

$$\int N_\epsilon e^{iS^0(q'',t''|q,t''-\epsilon)}f(q)drd\theta d\varphi\Psi(q)=\Psi(q'')(1+o(1))\,, \tag{I.10}$$

where $\Psi(q)$ is an arbitrary function of q. Using the action (I.4) one finds

$$S^0(q'',t''|q,t''-\epsilon)=(m/2\epsilon)[(r''-r)^2+r^2\{(\theta''-\theta)^2+(\varphi''-\varphi)^2\sin^2\theta\}]\times(1+o\{(q''-q)^3\}). \tag{I.11}$$

By means of the formula

$$\lim_{\epsilon\to 0}(2\pi i\epsilon)^{-1/2}\exp(ix^2/2\epsilon)=\delta(x)\,,$$

which is the essence of the method of stationary phase,[5] the integral in (I.10) may be evaluated with the result

$$N_\epsilon f(r,\theta,\varphi)\left(\frac{2\pi i\epsilon}{m}\right)^{3/2}\frac{1}{r^2\sin\theta}=1. \tag{I.12}$$

This shows that it is not consistent to put $f(r,\theta,\varphi)=1$, such that the measure $\mathfrak{D}q$ would be independent of the history $q(t)$, but that f has to be taken as[6]

$$f(r,\theta,\varphi)=r^2\sin\theta;\quad N_\epsilon=(2\pi i\epsilon/m)^{-3/2}. \tag{I.13}$$

[3] We use units such that $\hbar=c=16\pi G=1$. Greek indices μ, $\nu=0$, 1, 2, 3; Latin indices i, $k=1$, 2, 3. Signature $(+---)$.

[4] Actually, the transition amplitudes are distributions. In order to carry out the indicated manipulations properly one has to smear the equations with suitable test functions on both sides. Such an operation will always be understood.

[5] See Appendix 4.

[6] Of course, f and N_ϵ are only determined up to a constant $N_\epsilon\to c^{-1}N_\epsilon$, $f\to cf$.

This result could of course have been derived by transforming the measure in Cartesian coordinates, $\prod_{i=1}^{N} dx_i dy_i dz_i$ to spherical coordinates; but there is no analogous transformation at hand to determine the measure in the gravitational case.

The Measure for Gravity

In Sec. V we apply the above procedure to the gravitational transition amplitude with the result

$$\mathfrak{D}g = \text{Const.} \prod_L |\det g|^{-5/2} |\det \mathbf{g}| \prod_{\mu \leq \nu} dg_{\mu\nu}, \quad (\text{I.14})$$

valid for a lattice whose hypersurfaces (including τ'' and τ') are characterized by x^0= constant and curves x^k= constant. $\prod_L$ is a product over all points of the lattice, $\det g$ is the ordinary determinant of g and $\det \mathbf{g}$ is the determinant of the intrinsic metric of the hypersurfaces.

This result differs from the measures given by Misner,[7] Klauder,[8] Laurent,[9] and DeWitt.[10] How do we understand the fact that our measure is not invariant under coordinate transformations? The reason is not to be found in our choice of the action, but in the very fact that we are dealing with a lattice to define the amplitude. Unless one finds a way to define the amplitude without making use of a lattice, he cannot expect the measure to be invariant. In the framework of a lattice the covariance of the amplitude has to be distinguished from independence of the choice of lattice, in which it is evaluated. As one sees from (I.13) the normalization constant depends on the spacing ϵ between the points of the lattice. The normalization constant will therefore be different for different lattices, even if the number of points of the two lattices are the same. The invariance argument given by Misner[7] consists of two steps. First, one renames the lattice points performing a coordinate transformation and then compares this lattice to a lattice whose points have the same coordinates in the old frame as the original points in the new frame. This clearly involves the comparison of two different lattices and therefore one has to account for a change of the normalization constant in the course of the argument which destroys its usefulness to find the measure. In other words, the lattice singles out a particular coordinate system. From now on we shall work in this particular coordinate system, where the family of hypersurfaces is characterized by

$$\tau_n: \quad x^0 = \tau_n = \tau' + n\epsilon = \text{const.} \qquad n = 0, 1, \cdots, N+1$$
$$\tau_0 = \tau', \qquad \tau_{N+1} = \tau''$$

(we use the same letter τ to denote the surface as well as its associated time) and the curves are given by x^i= constant.

II. THE ELECTROMAGNETIC FIELD

In order to illustrate the method we will use to quantize the gravitational field, let us consider the analogous steps in the simpler case of the electromagnetic field. The simplicity of this system is due to the fact that its gauge group is Abelian which implies the linearity of the field equations.[11] The quantization of the electromagnetic field in the framework of the Feynman sum-over-histories formalism has been investigated by Wheeler[12] and Laurent.[13] The formal expression for the probability amplitude reads

$$(A''\tau''|A'\tau') = \mathfrak{N} \int e^{iS} \mathfrak{D}A,$$
$$S = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} d^4x; \qquad (\text{II.1})$$
$$\mathfrak{D}A = \prod_L dA_0 dA_1 dA_2 dA_3.$$

A_μ denotes the vector potential and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the electromagnetic field strength. $\prod_L$ indicates a product over all points of the lattice.

Classical Action for Electromagnetic Field

The specification of the measure is not enough to define the integral completely. We also have to adopt a rule how to compute S as a function of the variables over which we integrate, i.e., as a function of the $A_\mu(x_L)$, where x_L is any point of the lattice. Feynman's original definition[14] specifies S as the sum of the contributions from the slices between successive surfaces of the lattice. The contribution from one slice is defined to be the value of S at the classical history $A_\mu^0(x)$ which satisfies the classical equations of motion and assumes the prescribed boundary values $A_\mu(x_L)$ on the two successive surfaces. We shall accept this definition in principle. However, we meet with the following difficulty characteristic of systems with a gauge group. Our Lagrangian is degenerate in the sense that the classical equations of motion for the potential $A_\mu^0(x)$ do not determine it uniquely. Moreover and more important, the value of the action at the classical path remains the same if we change the boundary conditions by a gauge transformation. Therefore, $\exp iS$ will be the same for histories which differ only by a gauge transformation. Since there are an infinite number of gauge equivalent histories, the integral will diverge. In order to overcome

[7] C. W. Misner, Rev. Mod. Phys. **29**, 497 (1957).
[8] J. R. Klauder, Nuovo Cimento **19**, 1059 (1961).
[9] B. Laurent, Arkiv Fysik **16**, 279 (1959).
[10] B. S. DeWitt, J. Math. Phys. **3**, 1073 (1962). Note however that the δ function occurring in the metric of the functional space leads to factors of the type $[\delta^4(0)]^{10}$ in the measure. Expressed in terms of lattice variables this is equivalent to $\Delta^{-30}\epsilon^{-10}$.

[11] R. Utiyama, Phys. Rev. **101**, 1597 (1956); M. Gell-Mann and S. L. Glashow, Ann. Phys. (N. Y.) **15**, 437 (1961).
[12] J. A. Wheeler, unpublished lecture notes, University of Leyden, 1956.
[13] B. Laurent, Nuovo Cimento **4**, 1445 (1956).
[14] R. P. Feynman, Rev. Mod. Phys. **20**, 267 (1948).

this difficulty, let us decompose each history into two parts $a_\mu(x)$ and $\Lambda(x)$ by[13]

$$A_\mu(x)=a_\mu(x)+\partial_\mu\Lambda(x)\,,\quad a_0(x)=0 \qquad \text{(II.2)}$$

and consider the fields $a_\mu(x)$ and $\Lambda(x)$ separately. a_μ describes the gauge-independent part of A_μ while Λ is affected by gauge transformations Σ in the simple manner $\bar{\Lambda}=\Lambda+\Sigma$. We fix the choice of the gauge-independent part a_μ by the condition $a_0=0$ and furthermore dispose of the arbitrariness in Λ with the help of the condition $\Lambda(x)=0$ for $x\epsilon\tau'$.

Let us first keep $\Lambda(x)$ fixed and apply Feynman's rule to the histories $a_\mu(x)$. The extremal history is characterized by

$$\Box a_k{}^e(x)-\partial_k\partial^l a_l{}^e(x)=0\,,\quad a_0{}^e(x)=0\,,\qquad k,l=1,2,3\,. \qquad \text{(II.3)}$$

Although it is not difficult to determine the solution $a_k{}^e(x)$ exactly in terms of given boundary values on τ_i and τ_{i+1}, we restrict ourselves to the following approximation.[15] We are interested in the extremal action only in the limit $\tau_{i+1}-\tau_i=\epsilon\to 0$. Clearly if we specify $a_k{}^{(i)}$ and $a_k{}^{(i+1)}$ on τ_i and τ_{i+1} arbitrarily, the time derivatives of the solution $a_k{}^e$ that connects these boundary values will blow up as $\epsilon\to 0$ while derivatives with respect to x^1, x^2, x^3 will tend to some average of those on τ_i and on τ_{i+1} and thus remain finite. To find the extremal history $a_k{}^e(x)$ as $\epsilon\to 0$ it is therefore sufficient to consider solutions with the properties

$$\partial_l a_k{}^e(x)=o(1)\,;\quad \partial_0 a_k{}^e(x)=o(1/\epsilon)\,. \qquad \text{(II.4)}$$

In this approximation (II.3) becomes

$$\ddot{a}_k{}^e(x)=0+o(1)\,.$$

Thus we have

$$\dot{a}_k{}^e(x)=(1/\epsilon)(a_k{}^{(i+1)}-a_k{}^{(i)})(1+o(\epsilon^2))\,,$$

and the expression for the action in terms of the boundary values $a_k{}^{(i)}$ and $a_k{}^{(i+1)}$ is

$$S^e{}_{i+1,i}=-\frac{1}{2\epsilon}\int(\mathbf{a}^{(i+1)}-\mathbf{a}^{(i)})^2d^3x(1+o(\epsilon^2))\,.$$

If we express this in terms of the original variables A_μ, by solving (II.2) for $a_k(x)$ and $\Lambda(x)$, we find

$$S^e{}_{i+1,i}=-\frac{1}{2\epsilon}\int[\mathbf{A}^{(i+1)}-\mathbf{A}^{(i)} -\nabla(\Lambda^{(i+1)}-\Lambda^{(i)})]^2d^3x(1+o(\epsilon^2))\,, \qquad \text{(II.5)}$$

$$\Lambda^{(i+1)}-\Lambda^{(i)}=\int_{\tau_i}^{\tau_{i+1}}A_0(x)dx^0\,.$$

This shows explicitly, that we may change the history $A_0(x)$ without changing the value of the action, if only we keep the integral $\Lambda^{(i+1)}-\Lambda^{(i)}$ fixed. The only condition we find for the stationarity of the action against variations of A_0 or equivalently of Λ is

$$\Delta(\Lambda^{(i+1)}-\Lambda^{(i)})=\nabla\cdot(\mathbf{A}^{(i+1)}-\mathbf{A}^{(i)})\,. \qquad \text{(II.6)}$$

If we let $\tau_{i+1}\to\tau_i$ this reduces to the fourth of Maxwell's equations

$$\Delta A_0=\nabla\cdot\partial_0\mathbf{A}\,,$$

which has the character of a subsidiary condition, since it involves only first-order time derivatives.

If we insert the condition (II.6) in (II.5), we find

$$S^0{}_{i+1,i}=-\frac{1}{2\epsilon}\int(\mathbf{B}^{(i+1)}-\mathbf{B}^{(i)})^2d^3x(1+o(\epsilon^2))\,, \qquad \text{(II.7)}$$

$$B_k(x)=A_k(x)-\partial_k\int G(\mathbf{x}-\mathbf{y})\partial_l A_l(\mathbf{y})d^3y\,,\qquad \Delta G(\mathbf{x})=\delta(\mathbf{x})\,. \qquad \text{(II.8)}$$

$S^0{}_{i+1,i}$ denotes the value of the action at the extremal of both a_k and Λ, or what is equivalent, both A_k and A_0. B_k is the gauge-invariant quantity formed out of A_k and satisfies $\partial_k B_k=0$.

Sum Over Histories for Electromagnetic Field

Let us return now to the quantum-mechanical system. Since the value of the action at the extremal histories is given by $S^0{}_{i+1,i}$, Feynman's rule states that the infinitesimal transition amplitude is given by $N_\epsilon\exp iS^0{}_{i+1,i}$. Consider its action on any given state functional. We have

$$\Psi'(A_\mu{}^{(i+1)})=\lim_{\epsilon\to 0}N_\epsilon\int e^{iS^0{}_{i+1,i}}\times\prod_{\tau_i}dA_0{}^{(i)}dA_1{}^{(i)}dA_2{}^{(i)}dA_3{}^{(i)}\Psi(A_\mu{}^{(i)})\,. \qquad \text{(II.9)}$$

Here $\prod_{\tau_i}$ denotes a product over those points of the lattice which lie on τ_i. Since $S^0{}_{i+1,i}$ is independent of $A_0{}^{(i+1)}$ and furthermore depends on $A_k{}^{(i+1)}$ only through the gauge-invariant combination $B_k{}^{(i+1)}$, Ψ' must also have this property. We write it as $\Psi'(A_k{}^{(i+1)})$ and note that

$$\Psi'(A_k+\partial_k\chi)=\Psi'(A_k)\,. \qquad \text{(II.10)}$$

This implies immediately, that as $\tau_{i+1}\to\tau_i$, in general, Ψ' will not converge to Ψ, since this functional will, in general, not have this property.[16] In other words, the infinitesimal amplitude does not reduce to a delta functional as $\tau_{i+1}\to\tau_i$; it reduces to a projection

[15] See Ref. 14.

[16] We discuss only the properties of the amplitude and for that matter Ψ is an arbitrary test functional. Of course if Ψ is a physical state functional, i.e., independent of A_0 and gauge invariant, then $\Psi'\to\Psi(\epsilon\to 0)$.

operator onto functionals independent of A_0 and which are gauge invariant in the sense (II.10). Another feature of (II.9) is that it displays explicitly the degeneracy in the integrations over A_μ. If, e.g., Ψ is independent of A_0 then the integration over A_0 clearly diverges, which implies that the normalization constant vanishes. In other words, we have to *average* over A_0 rather than to integrate in order to keep N_ϵ well defined. However, there is still a remaining degeneracy left in the variables A_1, A_2, A_3, since $S^0{}_{i+1,i}$ depends only on $B_k{}^{(i)}$. By virtue of $\partial_k B_k=0$ these are only two independent variables. If we put

$$A_k=B_k+\partial_k C;\quad \partial_k A_k=\Delta C, \qquad \text{(II.11)}$$

and insert this nonsingular linear transformation in the measure and at the same time restrict the integrations on B_k to the infinite-dimensional plane $\partial_k B_k=0$ with the help of a delta functional, we obtain

$$\Psi'(A_k{}^{(i+1)}) = \lim_{\tau_{i+1}\to\tau_i} N_\epsilon' \int \exp\left[iS_{i+1,i}{}^0+i\int D(\mathbf{x})\partial_k B_k{}^{(i)}(\mathbf{x})d^3x\right] \times\prod_{\tau_i} dA_0{}^{(i)}dB_1{}^{(i)}dB_2{}^{(i)}dB_3{}^{(i)}dCdD \times\Psi(B_k{}^{(i)}+\partial_k C, A_0{}^{(i)}). \qquad \text{(II.12)}$$

In this form the divergencies are entirely contained in the integrations over $A_0{}^{(i)}$ and C which have simply to be replaced by averages in order to keep the normalization constant finite.

Gauge Invariance of the State Functional

Applying the method of stationary phase, which will be discussed in some detail in the gravitational case, one finds from (II.12)

$$\Psi'(A_k)=\int\Psi(A_k+\partial_k C, A_0)\prod_{\tau_i} dCdA_0 \Big/ \int\prod_{\tau_i} dCdA_0, \qquad \text{(II.13)}$$

if the normalization constant is appropriately chosen. This displays explicitly the gauge invariance of Ψ', since the right-hand side is an average over all gauges.

The property (II.10) may be trivially obtained from the formal expression (II.1). The reason we went through this detailed discussion at all is only because its analog in the gravitational case cannot be obtained in a satisfactory way by means of formal manipulations. This is because the analogous equations to (II.10) arise from invariance under transformations of the coordinate system. If one wants to derive any formal consequences of this invariance from the analog of (II.1), he has to change the lattice as well as the coordinate system, which makes the derivation of these conditions very clumsy.

Formal Derivation of Gauge Invariance

Consider the transformation

$$A_\mu\to\bar{A}_\mu=A_\mu+\partial_\mu\Sigma. \qquad \text{(II.14)}$$

Since the measure as well as the exponential are invariant under such a transformation, we expect to have

$$(A_\mu''+\partial_\mu\Sigma'', \tau''|A_\mu'+\partial_\mu\Sigma', \tau') = (A_\mu''\tau''|A_\mu'\tau'). \qquad \text{(II.15)}$$

Consider the special case $\Sigma'=\partial_0\Sigma'=0$. The two functions $f_1(\mathbf{x})=\partial_0\Sigma(\mathbf{x},\tau'')=\partial_0\Sigma''(\mathbf{x})$ and $f_2(\mathbf{x})=\Sigma(\mathbf{x},\tau'')=\Sigma''(\mathbf{x})$ may be chosen independently. For $f_2=0$ one obtains in particular

$$(A_0''+f_1, A_k'', \tau''|A_0', A_k', \tau') = (A_0'',A_k'',\tau''|A_0',A_k',\tau').$$

Since f_1 is arbitrary this implies that the amplitude is independent of A_0''. On the other hand, (II.15) is identical with (II.10) if $f_1=0$ and $f_2=\chi$.

Subsidiary Condition as Differential Form of Gauge Invariance

Let us discuss still another more interesting derivation. Consider again the transformation (II.14) this time with Σ infinitesimal. We have

$$S(\bar{A})=S(A)+\int\partial_\mu F^{\mu\nu}\partial_\mu\Sigma d^4x-\int_{\tau''-\tau'}\partial_\mu F^{\mu\nu}\Sigma d\sigma_\nu.$$

Again using the translational invariance of the measure this implies

$$\left(A''\tau''\middle|\int\partial_\mu F^{\mu\nu}\partial_\nu\Sigma d^4x -\int_{\tau''-\tau'}\partial_\mu F^{\mu\nu}\Sigma d\sigma_\nu\middle|A'\tau'\right)=0. \qquad \text{(II.16)}$$

By virtue of the field equations in matrix form

$$(A''\tau''|\partial_\mu F^{\mu\nu}(x)|A'\tau')=0, \qquad \text{(II.17)}$$

which may also be derived from the translational invariance of the measure,[17] we conclude

$$(A''\tau''|\partial_\mu F^{\mu 0}(x)|A'\tau')=0\quad x\epsilon\tau''\quad\text{or}\quad x\epsilon\tau'. \qquad \text{(II.18)}$$

Since the amplitude is independent of A_0', A_0'', let us for simplicity assume $A_0''=0$. Then for $x\epsilon\tau''$ this equation reduces to

$$(A''\tau''|\partial_k\dot{A}_k(x)|A'\tau')=0\quad x\epsilon\tau''. \qquad \text{(II.19)}$$

It may be shown that the matrix element of $\dot{A}_k$ is given

[17] Consider the more general translation $\bar{A}_\mu=A_\mu+\alpha_\mu$ and apply the same argument. See Ref. 7.

by

$$(A''\tau''|A_k(x)|A'\tau')$$
$$= -i\frac{\delta}{\delta A_k''(\mathbf{x})}(A''\tau''|A'\tau'), \quad x\epsilon\tau'', \quad \text{(II.20)}$$

as may be expected from the canonical formalism. Thus, the condition (II.19) amounts to

$$\partial_k[\delta/\delta A_k''(\mathbf{x})](A''\tau''|A'\tau')=0, \quad \text{(II.21)}$$

which is the differential form of (II.10). We shall refer to this condition as the subsidiary condition, since it corresponds exactly to the subsidiary condition in the canonical formalism, where we have

$$\partial_k\dot{A}_k=0 \quad (A_0=0).$$

We would like to emphasize that neither the independence of the amplitude from A_0' and A_0'' nor the fact that the amplitude is a gauge-invariant functional of A_k' and A_k'' arise from a particular choice of the gauge. They are straightforward consequences of the definition of the amplitude.

The Concept of Reduced Amplitude

In view of the application to the gravitational field let us briefly define and discuss the notion of reduced amplitude. As a preliminary note that it is convenient to use the variables $a_\mu(x)$ and $\Lambda(x)$ directly as variables of integration in the Feynman integral (II.1). In order to sum over all histories A_μ, we may sum over all gauge-independent parts a_μ as well as over all gauges Λ. This amounts to a transformation of the measure

$$\mathfrak{D}A=\mathfrak{D}a\mathfrak{D}\Lambda,$$
$$\mathfrak{D}a=\prod_L da_1da_2da_3, \quad \mathfrak{D}\Lambda=\prod_L d(\partial_0\Lambda), \quad \text{(II.22)}$$
$$\mathfrak{N}_a\mathfrak{N}_\Lambda=\mathfrak{N}.$$

Let us define the reduced amplitude by

$$\langle a''\tau''|a'\tau'\rangle=\mathfrak{N}_a\int e^{iS}\mathfrak{D}a, \quad \text{(II.23)}$$

such that

$$(A''\tau''|A'\tau')=\mathfrak{N}_\Lambda\int\langle a''\tau''|A'\tau'\rangle\mathfrak{D}\Lambda. \quad \text{(II.24)}$$

On the right-hand side, a_k'' has of course to be expressed in terms of A_k'' and $\partial_k\Lambda''$. The advantage of this way of splitting up the summation is that the reduced amplitude is a well-defined object, since the classical equations of motion for the field $a_k(x)$ are not degenerate. The constant $\mathfrak{N}_a$ can be normalized such that the reduced amplitude satisfies the composition law analogous to (I.7). Furthermore, because of gauge invariance, the reduced amplitude is independent of the history $\Lambda(x)$. The only place this history shows up is through the boundary value a_k'', when we wish to express the original amplitude in terms of the reduced one. This boundary value is given by $A_k''-\partial_k\Lambda''$. Clearly only the boundary value Λ'' of $\Lambda(x)$ enters. To carry out the integrations over Λ in (II.24) we have to apply Feynman's rule to the propagation of the history Λ. This is trivial between any two surfaces in the interior of the lattice, because $\Lambda(x)$ does not enter at all in the infinitesimal amplitude for a_k there. The extremal of a constant is the constant itself. To keep the normalization constant $\mathfrak{N}_\Lambda$ finite, we replace the integration over Λ again by an average, i.e., we may simply disregard it. This however does not apply to the propagation of Λ from the surface nearest to τ'' to τ''. Let us explicitly write the integrations on the last surface. Dropping the average over Λ on all other points of the lattice we obtain

$$(A''\tau''|A'\tau')=N_aN_\Lambda\int e^{iS^e{}_{\tau'',\tau''-\epsilon}}\langle a,\tau''-\epsilon|a'\tau'\rangle \times\prod_{\tau''-\epsilon} da_1da_2da_3d\partial_0\Lambda. \quad \text{(II.25)}$$

Here the requirement that S^e be stationary with respect to the history Λ as well is not trivial. The result is exactly the infinitesimal propagator $N_a \exp iS^0{}_{\tau'',\tau''-\epsilon}$ we obtained earlier. Thus we find

$$(A''\tau''|A'\tau')=N_a\int e^{iS^0{}_{\tau'',\tau''-\epsilon}}\langle a,\tau''-\epsilon|a''\tau'\rangle \times\prod_{\tau''-\epsilon} da_1da_2da_3. \quad \text{(II.26)}$$

If one makes use of the result (II.13) he finds

$$(A''\tau''|A'\tau') =\int\langle A_k''+\partial_kC,\tau''|A_k'\tau'\rangle dC\Big/\int dC. \quad \text{(II.27)}$$

The average over A_0 drops out since the reduced amplitude is independent of A_0. There is actually no asymmetry in initial and final states in (II.27), because the reduced amplitude satisfies

$$\langle A_k''+\partial_kC,\tau''|A_k'+\partial_kC,\tau'\rangle=\langle A_k''\tau''|A_k'\tau'\rangle$$

due to the fact that the choice $a_0=0$ allows for time-independent gauge transformations.

III. HEURISTIC ANALYSIS OF THE SUBSIDIARY CONDITIONS FOR THE GRAVITATIONAL FIELD

In this section we want to exploit the analogy with the electromagnetic field. Some of the manipulations will be purely formal and will have to be justified as soon as we introduce a properly defined framework to compute the transition amplitude. However, for a first orientation of what we may expect to find in a proper evaluation of the Feynman integral, these manipulations will be instructive.

States are Functionals of the Metric Induced on a Space-Like Hypersurface

What is the analog of the statement that the electromagnetic amplitude is independent of A_0' and A_0''? The reason for this was that in the gauge transformation (II.14) A_0 is transformed by the slope $\partial_0\Sigma$ which may be given arbitrarily on τ' and τ'', in contrast to the gradient $\partial_i\Sigma$ whose three components are of course not independent of each other.

Consider the analogous transformation

$$g_{\mu\nu}(x)=\partial_\mu\Lambda^\alpha\partial_\nu\Lambda^\beta\bar{g}_{\alpha\beta}(\Lambda(x)). \qquad \text{(III.1)}$$

Since we may choose the slopes $\partial_0\Lambda^\alpha$ arbitrarily, we may, in particular, take them such that $\bar{g}_{i0}=0$, $\bar{g}_{00}=1$ on τ' and on τ''. If we do not change the coordinate system inside τ' and τ'', we have, furthermore, $g_{ik}=\bar{g}_{ik}$ on τ' and on τ''. Near τ'' a transformation with these properties is given by

$$\begin{aligned}\Lambda^i(x)&=x^i-(g^{i0\prime\prime}/g^{00\prime\prime})(x^0-\tau''),\\ \Lambda^0(x)&=x^0+[(g^{00\prime\prime})^{-1/2}-1](x^0-\tau'')\end{aligned} \qquad \text{(III.2)}$$

and analogously near τ'. How does the action behave under such a transformation? With the help of the identity

$$\mathcal{L}=-R|g|^{1/2}+\partial_\lambda\{|g|^{1/2}(g^{\mu\nu}\Gamma_{\mu\nu}{}^\lambda-g^{\mu\lambda}\Gamma_{\mu\sigma}{}^\sigma)\}, \qquad \text{(III.3)}$$

where R is the curvature scalar, one finds

$$\begin{aligned}\int\mathcal{L}(g)d^4x&=\int\mathcal{L}(\bar{g})d^4\Lambda+\varphi_{\tau''}-\varphi_{\tau'},\\ \varphi_\tau&=\int_\tau|g|^{1/2}\partial_i\left(\frac{g^{i0}}{g^{00}}\right)g^{00}d^3x.\end{aligned} \qquad \text{(III.4)}$$

Since the action differs from $\int R|g|^{1/2}d^4x$ only by a surface term, only the boundary values of the transformation $\Lambda^\mu(x)$ occur in the transformation law, which are given by (III.2) in terms of g^{i0} and g^{00}.

Instead of summing over all histories $g_{\mu\nu}$ with the fixed boundary values $g_{\mu0}'$, $g_{\mu0}''$ we may as well sum over all histories $\bar{g}_{\mu\nu}$ which have orthogonal boundary values $\bar{g}_{\mu0}'=\bar{g}_{\mu0}''=\delta_\mu{}^0$ if only we account for the additional term in (IV.4). This shows that although the amplitude is not independent of $g_{\mu0}'$ and $g_{\mu0}''$ these variables are entirely contained in the phase factors $\exp i\varphi_{\tau''}$ and $\exp -i\varphi_{\tau'}$. Put

$$\mathfrak{N}\int e^{iS}\mathfrak{D}g=e^{i\varphi_{\tau''}}(\mathbf{g}''\tau''|\mathbf{g}'\tau')e^{-i\varphi_{\tau'}}, \qquad \text{(III.5)}$$

where the amplitude $(\mathbf{g}''\tau''|\mathbf{g}'\tau')$ now is a functional of the spatial components of the metric tensor on τ' and τ'' only. The unitary transformation $\exp i\varphi_\tau$ on all state vectors and amplitudes is trivial and we shall in the following deal always with the transformed amplitude $(\mathbf{g}''\tau''|\mathbf{g}'\tau')$. This corresponds exactly to the transition $\mathcal{L}\rightarrow\mathcal{L}^*$ in Dirac's Hamiltonian theory.[18] Thus, we may restrict ourselves to orthogonal boundary values $g_{\mu0}''=g_{\mu0}'=\delta_\mu{}^0$ on τ' and τ''.

Subsidiary Conditions as Differential Form of Coordinate Invariance

What may we expect to find as subsidiary conditions in the gravitational case? We have already stressed the fundamental difference between the gauge groups of electromagnetism and gravity. The electromagnetic gauge group leaves the structure of the lattice unaffected while the gravitational gauge group is the group of coordinate transformations and therefore affects the lattice as well as the field. Despite this difference, let us briefly sketch the formal analogy between electromagnetic and gravitational subsidiary conditions. In Sec. V we shall give a detailed derivation of the gravitational subsidiary conditions which accounts for the modifications due to the effect of the gauge group on the lattice. As a first orientation the formal analogy is instructive.

In analogy to the infinitesimal electromagnetic gauge transformation used in the derivation of the electromagnetic subsidiary condition at the end of Sec. II, let us consider an infinitesimal coordinate transformation

$$\begin{aligned}g_{\mu\nu}(x)\rightarrow\bar{g}_{\mu\nu}(x)&=g_{\mu\nu}(x)+\delta g_{\mu\nu},\\ \delta g_{\mu\nu}&=\nabla_\mu\delta\Lambda_\nu+\nabla_\nu\delta\Lambda_\mu.\end{aligned} \qquad \text{(III.6)}$$

Here ∇_μ denotes the covariant derivative with respect to the metric $g_{\mu\nu}$. Again making use of (III.3) one finds the transformation law for the action:

$$\begin{aligned}\delta S&=\int\mathcal{L}(\bar{g})d^4x-\int\mathcal{L}(g)d^4x\\ &=2\int_{\tau''-\tau'}S_\mu{}^\nu\delta\Lambda^\mu|g|^{1/2}d\sigma_\nu+\Delta,\\ S_\mu{}^\nu&=R_\mu{}^\nu-\tfrac{1}{2}\delta_\mu{}^\nu R,\\ \Delta&=\int_{\tau''-\tau'}[\Gamma_{\mu\nu}{}^\lambda-\delta_\nu{}^\lambda\Gamma_{\mu\alpha}{}^\alpha\\ &\qquad-\tfrac{1}{2}g_{\mu\nu}(g^{\alpha\beta}\Gamma_{\alpha\beta}{}^\lambda-g^{\alpha\lambda}\Gamma_{\alpha\beta}{}^\beta)]\delta g^{\mu\nu}|g|^{1/2}d\sigma_\lambda.\end{aligned} \qquad \text{(III.7)}$$

Since $g_{\mu0}=\delta_\mu{}^0$ on τ' and τ'', Δ becomes

$$\begin{aligned}\Delta&=\Delta_1+\Delta_2,\\ \Delta_1&=\frac{1}{2}\int_{\tau''-\tau'}(g^{lm}\dot{g}_{lm}g_{ik}-\dot{g}_{ik})\delta g^{ik}|g|^{1/2}d^3x,\\ \Delta_2&=-\frac{1}{2}\int_{\tau''-\tau'}\partial_ig_{lm}g^{lm}\delta g^{i0}|g|^{1/2}d^3x.\end{aligned} \qquad \text{(III.8)}$$

The origin of the term Δ_2 is obvious. The transformed

[18] P. A. M. Dirac, Proc. Roy. Soc. (London) **A246**, 333 (1958).

histories $\bar{g}_{\mu\nu}(x)$ do not in general have the properties $\bar{g}_{\mu 0}=\delta_\mu{}^0$ on τ' and τ''. Therefore we have to perform an infinitesimal unitary transformation $\exp i\delta\varphi_\tau$ on each of the two surfaces in order to find the transformed amplitude $(\bar{\mathbf{g}}''\tau''|\bar{\mathbf{g}}'\tau')$. This exactly cancels Δ_2. We thus have

$$\begin{aligned}(\bar{\mathbf{g}}''\tau''|\bar{\mathbf{g}}'\tau') &= \mathfrak{N}\int \exp\left(iS+2i\int S_\mu{}^\nu|g|^{1/2}\delta\Lambda^\mu d\sigma_\nu+i\Delta_1\right)\mathfrak{D}g \\ &= (\mathbf{g}''\tau''|\mathbf{g}'\tau') \\ &\quad +i\left(\mathbf{g}''\tau''\left|2\int S_\mu{}^\nu|g|^{1/2}\delta\Lambda^\mu d\sigma_\nu+\Delta_1\right|\mathbf{g}'\tau'\right). \qquad \text{(III.9)}\end{aligned}$$

On the other hand,

$$\begin{aligned}(\bar{\mathbf{g}}''\tau''|\bar{\mathbf{g}}'\tau') &= (\mathbf{g}''\tau''|\mathbf{g}'\tau')+\int\left\{\delta g_{ik}''(\mathbf{x})\frac{\delta}{\delta g_{ik}''(\mathbf{x})}\right. \\ &\quad \left.+\delta g_{ik}'(\mathbf{x})\frac{\delta}{\delta g_{ik}'(\mathbf{x})}\right\}(\mathbf{g}''\tau''|\mathbf{g}'\tau')d^3x. \qquad \text{(III.10)}\end{aligned}$$

Let us look closer at the quantity Δ_1. It is a linear functional of the velocities $\dot{g}_{ik}$, which we may express in terms of the canonical momenta

$$p^{ik}=\frac{\partial\mathfrak{L}}{\partial\dot{g}_{ik}}=\tfrac{1}{2}|g|^{1/2}(g^{il}g^{km}-g^{ik}g^{lm})\dot{g}_{ik}, \qquad (g_{\mu 0}=\delta_\mu{}^0). \qquad \text{(III.11)}$$

The quantity Δ_1 may be written

$$\Delta_1=\int_{\tau''}p^{ik}\delta g_{ik}''d^3x-\int_{\tau'}p^{ik}\delta g_{ik}'d^3x. \qquad \text{(III.12)}$$

Anticipating the result that the matrix element of $p^{ik}(x)$ with $x\epsilon\tau''$ is essentially a functional derivative with respect to its canonically conjugate coordinate g_{ik},

$$\begin{aligned}(\mathbf{g}''\tau''|p^{ik}(x)|\mathbf{g}'\tau') &= -i[\delta/\delta g_{ik}''(\mathbf{x})](\mathbf{g}''\tau''|\mathbf{g}'\tau'), \quad x\epsilon\tau'' \\ &= +i[\delta/\delta g_{ik}'(\mathbf{x})](\mathbf{g}''\tau''|\mathbf{g}'\tau'), \quad x\epsilon\tau' \qquad \text{(III.13)}\end{aligned}$$

which we shall justify later (we disregard factor ordering ambiguities at the moment), we see that Δ_1 cancels exactly the terms arising from the transformation of $\mathbf{g}'$ and $\mathbf{g}''$ in (III.10). Thus the subsidiary conditions read

$$(\mathbf{g}''\tau''|S_\mu{}^0|g|^{1/2}(x)|\mathbf{g}'\tau')=0, \quad x\epsilon\tau', \text{ or } x\epsilon\tau''. \qquad \text{(III.14)}$$

This is the analog of the electromagnetic subsidiary condition (II.12). We shall see in Sec. VI that these equations have to be modified slightly in order to account for the transformation properties of the measure.

The next step which consists of converting this subsidiary condition into differential form is considerably more complicated than the analogous step in the electromagnetic case, due to the nonlinearity of the gravitational system; this will also be carried out in Sec. V.

Subsidiary Conditions and Bianchi Identities

We would like to discuss briefly two other ways to arrive at the subsidiary conditions. The first one makes use of the Bianchi identities which imply

$$\int S_\mu{}^\nu\nabla_\nu\delta\Lambda^\mu d^4x=\int_{\tau''-\tau'}S_\mu{}^\nu\delta\Lambda^\mu d\sigma_\nu. \qquad \text{(III.15)}$$

The matrix element on the left-hand side vanishes if the field equations in matrix form hold, leaving us again with (III.14). This procedure is not strictly valid because (a) $\nabla_\mu\delta\Lambda^\nu$ in (III.15) involves $g_{\mu\nu}$ itself such that the field equations are not sufficient to show that the left-hand side vanishes and (b) Misner's method[19] to derive the field equations is not applicable to our noninvariant form of the measure.

Coordinate Invariance of the State Functionals

The second way is to argue directly from the gauge invariance of the amplitude under transformations which leave the surfaces τ' and τ'' invariant. A change of the coordinate system inside the surfaces τ' and τ'' leaves the amplitude unchanged. This implies that the amplitude depends only on the intrinsic geometries of these surfaces and not on the components of the metric tensor directly, which are of course affected by the transformations. This argument has to be considered as heuristic since only later will be in a position to prove the invariance of the amplitude under these transformations, which is equivalent to the subsidiary conditions (III.14) for $\mu=1, 2, 3$.

The fourth condition is associated with the vanishing of the Hamiltonian and cannot be obtained by this kind of argument. It involves coordinate transformations which change the shape of the surfaces τ' and τ''. Therefore we find a relation between two different amplitudes rather than one which states a property of the amplitude we are dealing with. This relation is the dynamical law for the transition amplitude.

For the particular case that the coordinate transormation under consideration carries the surfaces τ' and τ'' again into surfaces $x^0=$constant we obtain the result that the amplitude is independent of τ' and τ''.

IV. THE REDUCED AMPLITUDE

In this section we shall develop a method to define the Feynman amplitude for the gravitational field. Although no explicit expression for this amplitude is

[19] C. W. Misner, Rev. Mod. Phys. 29, 497 (1957).

obtained, the framework set up in this section will enable us to compute matrix elements between two states, each defined on the intrinsic geometry of a space-like hypersurface.

Gaussian Coordinates

We proceed in close analogy with the discussion of the amplitude for the electromagnetic field. As an analog of (II.2) we take

$$g_{\mu\nu}(x)=\partial_\mu\Lambda^\alpha(x)\partial_\nu\Lambda^\beta(x)a_{\alpha\beta}(\Lambda(x))\,;\quad a_{\mu 0}\equiv\delta_\mu{}^0, \qquad \text{(IV.1)}$$

i.e., we introduce a Gaussian coordinate system. It is known that a nonsingular transformation $\Lambda^\mu(x)$ leading to a Gaussian coordinate system exists in a finite region of space-time. However the extension of this region depends on the behavior of $g_{\mu\nu}(x)$. One may easily construct sets of four-geometries, which are nonsingular in a common domain, but do not admit Gaussian coordinates in any common region. Suppose, e.g., that $g_{\mu\nu}(x)$ is nonsingular and admits a Gaussian coordinate system throughout the region between $x^0=0$ and $x^0=T$, but not for $x^0<0$ or $x^0>T$. Define a set of four-geometries $g_{\mu\nu}{}^{(\lambda)}(x)$ by

$$g_{\mu\nu}{}^{(\lambda)}(x)=\lambda^2 g_{\mu\nu}(\lambda x)\,;\quad \lambda\geq 1.$$

These geometries admit a Gaussian coordinate system in the interval $0\leq x^0\leq T/\lambda$. If the set includes geometries $g_{\mu\nu}{}^{(\lambda)}(x)$ with arbitrarily high λ then there is no common region where each of the geometries admits a Gaussian coordinate system.

This implies that for however small we choose the spacing between τ' and τ'', there are always nonsingular histories $g_{\mu\nu}(x)$ which do not admit a Gaussian coordinate system, nonsingular throughout the region between τ' and τ''. Our attitude towards these difficulties will be to restrict the summation to those $g_{\mu\nu}(x)$ which do admit such a coordinate system throughout the region between τ' and τ''. We shall restrict the summation furthermore to those histories $g_{\mu\nu}(x)$ for which all the surfaces τ_i of the lattice are totally space-like. This is already implied in the form (I.14) of the measure. Although these might seem to be enormous restrictions, we would like to point out, that the subsidiary conditions we want to obtain and which are the heart of the matter are of differential character. Furthermore they concern only the boundaries τ' and τ'' of the domain of the histories $g_{\mu\nu}(x)$.

We complete the specification of Λ^μ and a_{ik} by taking $\Lambda^\mu(x)=x^\mu(x\epsilon\tau')$. This implies $a_{ik}'=g_{ik}'$.

Transformation of the Measure to New Variables of Integration

Instead of summing over all histories $g_{\mu\nu}(x)$ we may equally well sum over all histories $a_{ik}(\Lambda)$ and over all transformations to Gaussian coordinate systems $\Lambda^\mu(x)$. The boundary values of these histories at τ' and τ'' have to be chosen such that the resulting boundary values for $g_{\mu\nu}$ satisfy $g_{\mu 0}'=g_{\mu 0}''=\delta_\mu{}^0$. The Jacobian of this transformation of the variables of integration is computed in Appendix 1 for a measure of the form[20]

$$\mathfrak{D}_{\alpha\beta}g=\prod_L M|\det g|^\alpha|\det \mathbf{g}|^\beta\prod_{\mu\leq\nu}dg_{\mu\nu}, \qquad \text{(IV.2)}$$

which contains the two parameters α and β. (We shall work with this general measure and later determine the values of α and β to be $-\frac{5}{2}$ and 1 with the help of the consistency argument given in Sec. I.) The result we find in Appendix 1 reads

$$\prod_{\mu\leq\nu}dg_{\mu\nu}=2|\det\tilde{\Lambda}|^4|\det\Lambda|\times|\det a|\prod_{i\leq k}da_{ik}\prod_\mu d(\partial_0\Lambda^\mu), \qquad \text{(IV.3)}$$

where $\det\tilde{\Lambda}$ denotes the determinant of $\partial_i\Lambda^k(i,k=1,2,3)$. The 10 variables of integration $g_{\mu\nu}$ are thereby replaced by the 6 variables a_{ik} and 4 variables $\partial_0\Lambda^\mu$. With the further notation

$$g_{ik}=\partial_i\Lambda^l\partial_k\Lambda^m a_{lm}+\partial_i\Lambda^0\partial_k\Lambda^0=\partial_i\Lambda^l\partial_k\Lambda^m b_{lm}, \qquad \text{(IV.4)}$$

$$\mathfrak{D}_{\alpha\beta}a=\prod_L M|\det\tilde{\Lambda}|^{2\beta+4}|\det\Lambda|^{2\alpha+2}\times|\det a|^{\alpha+1}|\det b|^\beta\prod_{i\leq k}da_{ik}, \qquad \text{(IV.5)}$$

$$\mathfrak{D}\Lambda=\prod_L|\det\Lambda|^{-1}\prod_\mu d(\partial_0\Lambda^\mu),$$

we have

$$\mathfrak{D}_{\alpha\beta}g=\mathfrak{D}_{\alpha\beta}a\mathfrak{D}\Lambda\,;\quad 2\mathfrak{N}=\mathfrak{N}_a\mathfrak{N}_\Lambda. \qquad \text{(IV.6)}$$

Transformation of the Action

The action transforms according to

$$\int\mathfrak{L}(g)d^4x=\int\mathfrak{L}(a)d^4\Lambda+\chi_{\tau''}-\chi_{\tau'},$$

$$\chi_\tau(\Lambda,\mathbf{g})=-\int_\tau(g^{\rho\sigma}\partial_\sigma\Lambda_\mu{}^\nu\Lambda_\nu{}^{-1\mu}-g^{\sigma\mu}\partial_\sigma\Lambda_\mu{}^\nu\Lambda_\nu{}^{-1\rho})|g|^{1/2}d\sigma_\rho. \qquad \text{(IV.7)}$$

Because a coincides with $\mathbf{g}$ on τ' by construction the surface term $\chi_{\tau'}$ vanishes. Although the expression above for $\chi_{\tau''}$ contains also terms like $\partial_0\Lambda^\mu$ (the terms $\partial_{00}\Lambda^\mu$ cancel), these may be expressed in terms of g_{ik}'' and $\Lambda^{\mu\prime\prime}(\mathbf{x})=\Lambda^\mu(\mathbf{x},\tau'')$. On the other hand the reduced action, $\int\mathfrak{L}(a)d^4\Lambda$, is independent of the history $\Lambda^\mu(x)$. It is therefore convenient to introduce the reduced amplitude

$$\langle a''\tau''|a'\tau'\rangle_\Lambda=\mathfrak{N}_a\int\exp\left[i\int\mathfrak{L}(a)d^4\Lambda\right]\mathfrak{D}_{\alpha\beta}a. \qquad \text{(IV.8)}$$

[20] For reasons which will become clear later, we include part of the normalization constant in $\mathfrak{D}_{\alpha\beta}g$.

With this notation the original amplitude may be written

$$(\mathbf{g}''\tau''|\mathbf{g}'\tau')=\mathfrak{N}_\Lambda\int e^{i\chi_{\tau''}(\Lambda'',\mathbf{g}'')}\langle a''\tau''|\mathbf{g}'\tau'\rangle_\Lambda\mathfrak{D}\Lambda. \quad \text{(IV.9)}$$

In this formula a'' has of course to be expressed in terms of g_{ik}'' and $\Lambda^{\mu\prime\prime}$.

The essential point in this way of splitting up the integrations over $g_{\mu\nu}$ is that the reduced amplitude is now a well-defined object, since the Lagrangian $\mathfrak{L}(a)$ is not degenerate. For any given fixed history $\Lambda^\mu(x)$ we may perform the limiting operations in the Feynman sum-over-histories prescription as applied to the histories a_{ik}. $\Lambda^\mu(x)$ determines the location of the lattice points in Λ space as well as the shape of the surface τ''. The resulting object $\langle a''\tau''|a'\tau'\rangle_\Lambda$ may then be inserted in (IV.9) to carry out the Λ integrations.

The Special Case $\Lambda^\mu(x)=x^\mu$

(a) *Approximation for Histories Connecting Nearby Surfaces*

Let us discuss the simplest case first, where $\Lambda^\mu(x)=x^\mu$, i.e., the lattice in Λ space is identical to the one we started with, consisting of equally spaced hypersurfaces $\Lambda^0=\tau_n=\tau'+n\epsilon$ and straight lines $\Lambda^i=$constant.

As a first step in the construction of the reduced amplitude we have to compute the extremal action for the histories $a_{ik}(\Lambda)$. Fortunately, we are not interested in the general case of prescribed boundary values on two arbitrary surfaces, but only in the limit of small surface spacing ϵ. Denote the boundary values on $\tau_i=\tau_-$ by a_{ik}^-, those on $\tau_{i+1}=\tau_+$ by a_{ik}^+. It is clear that as the spacing ϵ tends to zero, time derivaties will blow up, being essentially determined by $(1/\epsilon)(a_{ik}^+-a_{ik}^-)$, while the derivatives with respect to Λ^1, Λ^2, Λ^3 will tend to some average of those on τ_- and those on τ_+ and thus remain finite. In other words we are interested in a solution of the equations of motion with the properties[21]

$$\partial_i a_{lm}=o(1)\,,\quad \partial_0 a_{lm}=o(1/\epsilon)\,. \qquad \text{(IV.10)}$$

(b) *The Equations of Motion*

The equations resulting from the Lagrangian $\mathfrak{L}(a)$ read

$$\begin{aligned}R_{ik}(a)-\tfrac{1}{2}a_{ik}R(a)&=R_{ik}^{(3)}-\tfrac{1}{2}a_{ik}R^{(3)}-\tfrac{1}{2}\ddot{a}_{ik}\\&\quad+\tfrac{1}{2}a_{ik}a^{lm}\ddot{a}_{lm}-\tfrac{1}{4}\dot{a}_{ik}a^{lm}\dot{a}_{lm}+\tfrac{1}{2}\dot{a}_{il}a^{lm}\dot{a}_{mk}\\&\quad+\tfrac{1}{8}a_{ik}(\dot{a}_{lm}a^{lm})^2-\tfrac{3}{8}a_{ik}a^{lm}a^{rs}\dot{a}_{lr}\dot{a}_{ms}=0\,; \qquad \text{(IV.11)}\end{aligned}$$

$\dot{a}_{ik}=\partial_0 a_{ik}$ and $R_{ik}^{(3)}=R^{(3)l}{}_{ikl}$ denotes the contracted curvature tensor of the hypersurface $\Lambda^0=$constant, formed with the metric a_{ik}. Thus, as a first approximation we may disregard the term $R_{ik}^{(3)}-\frac{1}{2}a_{ik}R^{(3)}$ being $o(1)$ as compared to $o(1/\epsilon^2)$ of the rest. We have to solve the resulting second-order ordinary differential equations exactly, since all terms have the same behavior as $\epsilon\to 0$. This is carried out in Appendix 2.

(c) *The Path is Unique*

An important result derived in Appendix 2 is that there is only one classical path connecting two given boundary values a_{ik}^+ and a_{ik}^- on τ_+ and τ_-, in the limit $\tau_+\to\tau_-$. This property of the reduced action—together with the fact that the action is quadratic in the velocities—justifies the sum over histories procedure in the sense that only for such actions is the Feynman quantization equivalent to the ordinary quantization.[22]

(d) *Solution in Terms of Eigenvalue Variables*

In Appendix 2 the value of the reduced action at the classical path is computed and expressed in terms of eigenvalue variables defined as follows: a_{ik}^+ and a_{ik}^- are negative definite by assumption ($\Lambda^0=$constant are space-like hypersurfaces). Therefore they may be simultaneously diagonalized.

$$a_{ik}^\pm=-\sum_{l=1}^{3}S_i{}^lS_k{}^lA_\pm{}^l. \qquad \text{(IV.12)}$$

To specify $S_i{}^l$ uniquely we choose $\det S=1$ and

$$A_-{}^l=|\det a_-|^{1/3},\quad l=1,2,3. \qquad \text{(IV.13)}$$

Furthermore, put

$$\begin{aligned}A_+{}^l&=|\det a_+|^{1/3}\alpha^l,\\ \alpha&=[\tfrac{3}{32}\sum_l(\ln\alpha_l)^2]^{1/2},\quad A_\pm=|\det a^\pm|^{1/4}.\end{aligned} \qquad \text{(IV.14)}$$

Then the extremal value of the action is given by

$$\begin{aligned}S^e&=\int_{ex}\mathfrak{L}(a)d^4\Lambda\\&=-\frac{8}{3\epsilon}\int(A_+{}^2+A_-{}^2-2A_+A_-\cosh\alpha)d^3x+o(\epsilon)\,. \quad \text{(IV.15)}\end{aligned}$$

This formula is valid for the special case $\Lambda^\mu(x)=x^\mu$ and expresses the value of the action at the stationary history as a functional of the eigenvalues of a_{ik}^+ and a_{ik}^- in their simultaneous diagonal form.

(e) *Integrating Over the Gaussian Metric*

We shall deal with the general history $\Lambda^\mu(x)$ later. What we want to do first is to apply Feynman's definition of the infinitesimal amplitude to the especially simple expression for the extremal action just obtained.

[21] The situation is exactly the same as in the case of the infinitesimal propagator for a particle in a potential $V(x)$. See R. P. Feynman, Rev. Mod. Phys. **20**, 267 (1948).

[22] In this connection, see also P. Chocquard, Helv. Phys. Acta **28**, 89 (1955).

Let us investigate the action of the infinitesimal reduced amplitude, $\exp iS^e$, on an arbitrarily given test functional $\Psi(a)$, i.e., consider the integral

$$\Psi'(a^+)=\int\langle a^+\tau_+|a^-\tau_-\rangle_{\Lambda=x}\mathfrak{D}_{\alpha\beta}{}^{\tau-}a^-\Psi(a^-)$$

$$=N_a\int e^{iS^e(a^+,a^-)}\Psi(a^-)\mathfrak{D}_{\alpha\beta}{}^{\tau-}a^-. \qquad \text{(IV.16)}$$

Here, $\mathfrak{D}_{\alpha\beta}{}^{\tau}a$ is the same as $\mathfrak{D}_{\alpha\beta}a$ in (V.5) except that the product $\prod_L$ now runs only over the points of the surface τ instead of the whole lattice. According to our restrictions on the summation, we integrate only over negative definite boundary values a_{ik}^-. The limits of integration when expressed in terms of the variables a_{ik}^- themselves are very cumbersome, because the requirement of negative definiteness is a nonlinear system of inequalities in the variables a_{ik}^-. A much more convenient choice of the variables of integration is given by the eigenvalues we already used to compute the action. Of course the three eigenvalues $A_-{}^l$ defined in (IV.12) are not sufficient to replace the six variables a_{ik}^-. We have to integrate over the matrices $S_i{}^l$ in (V.12) as well. Since they satisfy

$$\sum_l S_i{}^lS_k{}^l=-a_{ik}{}^+A_+{}^{-4/3}, \quad \det S=1, \qquad \text{(IV.17)}$$

we see that they are the matrices of the rotation group in three dimensions associated with the metric $u_{ik}{}^+=-a_{ik}{}^+A_+{}^{-4/3}$, which is a three-parameter Lie group. We have in matrix notation

$$SS^t=u^+. \qquad \text{(IV.18)}$$

In order to be able to use the well-known results of the theory of the ordinary rotation group in 3 dimensions associated with the metric δ_{ik}, we write

$$S=TR \qquad \text{(IV.19)}$$

and choose the fixed matrix T such that it diagonalizes u^+

$$T^{-1}u^+(T^{-1})^t=1. \qquad \text{(IV.20)}$$

Then the matrix R satisfies

$$RR^t=1,$$

i.e., it is an element of the ordinary rotation group.

An extra benefit of this choice of the variables of integration is that the extremal action S^e is independent of R, such that the integration over the rotation group will be very simple.

(*f*) *Transformation of the Measure to Eigenvalue Variables*

We now have to transform the measure to our six new variables of integration, which we may choose to be given by $A_-{}^1$, $A_-{}^2$, $A_-{}^3$, γ_1, γ_2, γ_3, where γ_1, γ_2, γ_3 are the Euler angles parametrizing the rotation group. This transformation is carried out in Appendix 3 with the result

$$da_{11}{}^-\cdots da_{33}{}^-=d^3A^-d^3R,$$
$$d^3A^-=(A_-{}^3-A_-{}^2)(A_-{}^1-A_-{}^3)\times(A_-{}^2-A_-{}^1)dA_-{}^1dA_-{}^2dA_-{}^3, \qquad \text{(IV.21)}$$

$$d^3R=\sin\gamma_2 d\gamma_1 d\gamma_2 d\gamma_3. \qquad \text{(IV.22)}$$

The variables of the rotation group appear in the group invariant measure d^3R. This is a consequence of the invariance of the product of the six differentials on the left-hand side with respect to the linear transformations

$$\tilde{a}^-=Va^-V^t, \quad VV^t=1,$$

which rotate the axis of the coordinate frame in the space of the Euler angles, thus establishing rotational invariance of the right-hand side.

In order to establish a one-to-one correspondence between the variables a_{ik}^- on the one hand and $A_-{}^l$, $S_i{}^l$ on the other hand, let us order the eigenvalues according to $0<A_-{}^1\leq A_-{}^2\leq A_-{}^3<\infty$. This still does not fix the matrix S uniquely. As discussed in Appendix 3 there are four equivalent matrices S which satisfy these requirements. Thus if one integrates over all eigenvalues $A_-{}^l$ satisfying the above restrictions and all matrices S (the full rotation group) he obtains all negative definite matrices a_{ik}^- and every one exactly four times. Accounting for this degeneracy by a factor of 4 the limits of integration become very simple. We have to integrate over the full rotation group and over all $A_-{}^l$ satisfying the above restrictions. Still considering the special case $\Lambda^\mu(x)=x^\mu$, we have, in view of the fact that the exponential is independent of the Euler angles

$$\Psi'(a^+)=N_a\int e^{iS^e}\times\prod_{\tau-}M\tfrac{1}{4}|\det a^-|^{\alpha+\beta+1}d^3A^-\bar{\Psi}(A_-{}^l). \qquad \text{(IV.23)}$$

Here, $\bar{\Psi}$ denotes the integral of Ψ over the rotation group.

(*g*) *Asymptotic Expansion*

Fortunately, we do not need an exact evaluation of the integral but are only interested in an asymptotic expansion in ϵ when $\epsilon\to0$. The method of stationary phase is particularly suited for this purpose, because S^e has the property that considered as a function of $A_-{}^1$, $A_-{}^2$, $A_-{}^3$ it has only one stationary point[23]

$$\delta S^e/\delta A_-{}^l=0 \quad \text{for} \quad A_-{}^l=A_+{}^l. \qquad \text{(IV.24)}$$

The relevant estimates are given in Appendix 4. They

[23] This property can be understood directly without knowing the explicity solutions of the equations of motion. See the discussion in the case of the general history $\Lambda^\mu(x)$.

lead to the asymptotic expansion

$$\Psi'=\bar{\Psi}(A_+{}^l)N_a\prod_L M2\pi\cdot(\epsilon/\Delta^3)^3|\det a_+|^{\alpha+\beta+\frac{3}{2}}(1+o(\epsilon^{1/2})).\quad\text{(IV.25)}$$

The power $(\epsilon/\Delta^3)^3$ is reasonable, since we originally had six variables of integration a_{ik}^- each one giving rise to a factor $(\epsilon/\Delta^3)^{1/2}$. The exponent $\alpha+\beta+\frac{3}{2}$ comes about as follows. The action is invariant under a stretching of the coordinates $\Lambda^i\to\lambda\Lambda^i$, i.e., under the simultaneous change $a_{ik}^\pm\to\lambda^{-2}a_{ik}^\pm$, $\Delta\to\lambda\Delta$. The measure $\mathfrak{D}_{\alpha\beta}a$ in (IV.5) changes by a factor $\lambda^{-6(\alpha+\beta+1)-12}$, which is indeed the factor picked up by Ψ' if the same substitution is inserted in (IV.25). To evaluate $\bar{\Psi}$ we simply note that for $A_-{}^l=A_+{}^l$ we have $a_{ik}^+=a_{ik}^-$ independent of S. Therefore[24]

$$\bar{\Psi}=\Psi(a_{ik}^+)\int d^3R=8\pi^2\Psi(a_{ik}^+).\quad\text{(IV.26)}$$

(h) *The Reduced Amplitude Connecting Nearby Surfaces*

If we insert the asymptotic expansion (IV.25) in the definition of Ψ', Eq. (IV.16), we find

$$\int\langle a^+\tau_+|a^-\tau_-\rangle_{\Lambda=x}\mathfrak{D}_{\alpha\beta}{}^{\tau_-}a^-\Psi(a^-)=\Psi(a^+)N_a\prod_{\tau+}M16\pi^3(\epsilon/\Delta^3)^3|\det a^+|^{\alpha+\beta+\frac{3}{2}}\times(1+o(\epsilon^{1/2})).\quad\text{(IV.27)}$$

In the limit $\tau_+-\tau_-=\epsilon\to0$ the correction term $o(\epsilon^{1/2})$ vanishes and the right-hand side is of the form $\Psi(a^+)$. Note that the constants N_a and M depend on ϵ and Δ. These constants will be normalized later in such a fashion that $Z=1$; $\alpha+\beta+\frac{3}{2}=0$. Therefore in the limit as $\tau_+\to\tau_-$ the reduced amplitude for the transition from τ_- to τ_+ behaves like a δ functional. This behavior was to be expected and is common to all Feynman amplitudes associated with nondegenerate actions quadratic in the velocities.[25]

Note that this property is not shared by Feynman amplitudes associated with degenerate actions. In particular the electromagnetic amplitude for transiting from τ' to τ'' does not reduce to a δ functional as $\tau''\to\tau'$; instead this amplitude reduces to a projection operator onto gauge invariant state functionals, as may be seen from Eq. (II.27). Likewise the full gravitational transition amplitude reduces to a projection operator onto states satisfying the subsidiary conditions. The concept of reduced amplitude is useful precisely because it is associated with a nondegenerate action.

The General History $\Lambda^\mu(x)$

(a) *Approximation for Histories Connecting Nearby Surfaces*

What modifications do we have to expect if $\Lambda^\mu(x)\neq x^\mu$? In this case the surfaces of the lattice in Λ space will not have the shape $\Lambda^0=$ constant and the family of curves that fix the locations of the lattice points will not be given by $\Lambda^i=$ constant. The direction of these curves is given by $\partial_0\Lambda^\mu$. As $\epsilon\to0$ the derivatives of a_{ik} with respect to directions that are parallel to the surfaces of the lattice will tend to some average of the derivatives inside the surfaces τ_- and τ_+ of a_{ik}^- and a_{ik}^+ and remain finite. On the other hand those along the direction $\partial_0\Lambda^\mu$ will be $o(1/\epsilon)$. Thus the first approximation will be given by

$$\partial_0\Lambda^\lambda\partial_\lambda a_{ik}=o(1/\epsilon),\quad \partial_l\Lambda^\lambda\partial_\lambda a_{ik}=o(1),\quad\text{(IV.28)}$$

which for $\Lambda^\mu=x^\mu$ again reduce to (IV.10). (IV.28) is equivalent to

$$\partial_\lambda a_{ik}=\chi_\lambda a_{ik}'+o(1),\quad \chi_\lambda=\Lambda^{-1}{}_\lambda{}^0,\quad a_{ik}'=\partial a_{ik}/\partial x^0=\partial_0\Lambda^\lambda\partial_\lambda a_{ik}.\quad\text{(IV.29)}$$

(b) *Equations of Motion*

We have to insert this approximation into the equations of motion (IV.11) and again solve them with prescribed boundary values on τ_i and τ_{i+1}. This problem is considerably more complicated than the case $\Lambda^\mu=x^\mu$, because the quantities $\partial_\mu\Lambda^\nu$ show up[26] in the equations of motion. These equations may be simplified when expressed in terms of the quantity b_{ik} defined in (IV.4). In these variables the approximation (IV.29) leads to the following action functional

$$\int\mathfrak{L}(a)d^4\Lambda=\frac{1}{4}\int b^{ik}b^{lm}(b_{il}'b_{km}'-b_{ik}'b_{lm}')\times|b|^{1/2}Kd^4x+o(1),\quad\text{(IV.30)}$$

$$K=\frac{|\det\tilde{\Lambda}|^2}{|\det\Lambda|}\left|\frac{\det b}{\det a}\right|^{1/2}.$$

[24] The integration over the full rotation group extends over the intervals $0\leq\gamma_1<2\pi$; $0\leq\gamma_2<\pi$; $0\leq\gamma_3<2\pi$.

[25] If one carries out the asymptotic expansion of (IV.16) to higher orders in $\epsilon^{1/2}$, he finds that—using the normalization of the measure as given in (IV.37)—the next nonvanishing term has the form $-i\epsilon\int\mathfrak{H}d^3x$ with the same $\mathfrak{H}(x)$ as in (V.23)–(V.26). Thus, $\int\mathfrak{H}d^3x$ is the infinitesimal generator of the unitary transformation which describes the dynamical evolution of the reduced amplitude and should therefore be Hermitian in the measure (IV.37). The Hermiticity of $\mathfrak{H}(x)$ may be directly verified with (V.23)–(V.26). [Note that $-i(\delta/\delta g_{ik})$ is *not* Hermitian in the measure (IV.37).] One finds that Hermiticity determines the coefficient of $\mathfrak{H}_1$ uniquely and the coefficient given in (V.25) is indeed correct; the coefficient of $\mathfrak{H}_2$ is obviously not affected by the Hermiticity condition.

[26] Note that the history $\Lambda^\mu(x)$ is kept fixed when we perform the limit $\epsilon\to0$ in the reduced amplitude. Thus the quantities $\partial_\mu\Lambda^\nu$ appearing in the equations of motion for a_{ik} are to be considered as time-independent, the variation of $\partial_\mu\Lambda^\nu$ along the path being $o(\epsilon)$.

The variations of this functional with respect to b_{ik} produce the required approximation to the equations of motion. The appearance of $\partial_\mu\Lambda^\nu$ in these equations arises from the variations of $|\det a|$ in K.

Fortunately we are not so much interested in the general solution of these approximate equations of motion, but need only the value of the action at the extremal histories. Furthermore, as we have seen in the special case $\Lambda^\mu = x^\mu$, in the context of the Feynman sum over histories formalism, only the behavior of the action as a functional of the boundary values near the stationary point

$$\delta S^e(b^+,b^-)/\delta b_{ik}{}^+ = 0 \qquad \text{(IV.31)}$$

is relevant.

(c) *A General Property of Nondegenerate Actions which are Homogeneous Quadratic in the Velocities*

It is easy to see that the action—considered as a function of the boundary values—corresponding to a Lagrangian which is nondegenerate and homogeneous quadratic in the velocities, has only one stationary point. Consider a variation about a solution of the equations of motion $q_i(\tau)$ which connects q_i^- at τ_- to q_i^+ at τ_+.

$$S^e = \int \mathfrak{L}(q_i,\dot{q}_k)d\tau, \qquad \text{(IV.32)}$$

$$\delta S^e = \int \left(\frac{\partial\mathfrak{L}}{\partial q} - \frac{d}{dt}\frac{\partial\mathfrak{L}}{\partial \dot{q}_i}\right)\delta q_i d\tau + \frac{\partial\mathfrak{L}}{\partial \dot{q}_i}\delta q_i \Big|_{\tau-}^{\tau+}. \qquad \text{(IV.33)}$$

Since $q_i(\tau)$ satisfies the equations of motion, one obtains

$$\partial S^e/\partial q_i{}^+ = (\partial\mathfrak{L}/\partial\dot{q}_i)(\tau_+) = p^i(\tau_+). \qquad \text{(IV.34)}$$

If q_i^+ is to be a stationary boundary point, $\partial S/\partial q_i^+$ vanishes and therefore the momentum $p_i(\tau_+)$ must vanish too. Since $\mathfrak{L}$ is nondegenerate and quadratic in the velocities this implies $\dot{q}_i(\tau_+)=0$ and by virtue of the homogeneity $\dot{q}_i(\tau)\equiv 0$. This shows that

$$q_i^+ = q_i^- \qquad \text{(IV.35)}$$

is the only stationary point.

(d) *Asymptotic Expansion*

This result may be applied to the reduced action as given by (IV.30). Only boundary values near stationary points—defined by (IV.31)—contribute in the sum over histories and there is only one stationary point $b_{ik}^+ = b_{ik}^-$. It is not difficult to investigate a neighborhood of this point, which is all we need to apply the method of stationary phase. Apart from the factor K the results are identical to those we obtained for the special case $\Lambda^\mu(x)=x^\mu$. One finds instead of (IV.27)

$$\int \langle a^+\tau_+ | a^-\tau_-\rangle_\Lambda \mathfrak{D}_{\alpha\beta}{}^{\tau-} a^- \Psi(a^-)$$
$$= \Psi(a_+) N_a \prod_{\tau+} 16\pi^3(\epsilon/\Delta^3)^3 |\det a_+|^{\alpha+5/2} |\det b_+|^{\beta-1}$$
$$\times |\det\Lambda|^{2\alpha+5} |\det\tilde{\Lambda}|^{2\beta-2} \times (1+o(\epsilon^{1/2})). \qquad \text{(IV.36)}$$

This of course contains (IV.27) as a special case.

Determination of the Measure

The asymptotic expansion (IV.36) invites us to normalize the reduced amplitude by requiring

$$\alpha = -\tfrac{5}{2}; \quad \beta = 1; \quad N_a{}^{-1} = \prod_\tau M 16\pi^3(\epsilon/\Delta^3)^3 \qquad \text{(IV.37)}$$

and we will denote $\mathfrak{D}_{-\frac{5}{2},1}a$ by $\mathfrak{D}a$.

We shall now try to justify this choice. *A priori* there is no reason why we should have to normalize the reduced amplitude such that it approaches a delta functional when $\tau'' \to \tau'$, because the reduced amplitude has no physical significance.

(a) *General Properties of Feynman Integrals*

The two assumptions on which this normalization is based are general properties of Feynman integrals which are very reasonable, but have so far not been proved in sufficient generality to be applied to our case:

(1) The sequence of refinements of the lattice defined by $N\to\infty$, $\epsilon\to 0$ leads to a sequence of Feynman integrals that converge to a well-defined limit, for a sufficiently broad class of nondegenerate Lagrangians.

This assumption is the analog of the theorem which asserts the existence of the Riemann integral and has been proved, e.g., for the amplitude of a nonrelativistic particle in a repulsive potential.[27]

The second assumption is the analog of the statement that the value of the Riemann integral is independent of the particular choice of the subdivision of the interval of integration occuring in the Riemann sum.

(2) If two lattices and their sequences of refinements are connected by a one-to-one well-behaved mapping which reduces to the identity mapping in a neighborhood of τ' and τ'' then the associated limits of the Feynman integrals are the same.

In other words a deformation of the lattice in the interior does not affect the amplitude, although it of course affects both the measure and the infinitesimal amplitudes. In view of the close connection between Feynman integrals and partial differential equations[28]

[27] D. G. Babbit, J. Math. Phys. 4, 36 (1963).

[28] J. M. Gelfand and A. M. Jaglom, Fortschr. Physik 5, 517 (1957).

(more precisely the associated Schrödinger equations) this property is also related to the fact, that one may change the coordinates of a partial differential equation by transforming the coefficients appropriately; the solutions of the transformed equation are the transforms of the solutions of the original equation.

(b) *Uniqueness of the Measure*

We assume that a measure with these two properties exists and want to show first that it must be given by (IV.37). The argument is simple: The very definition of the reduced amplitude implies the composition law

$$\langle a''\tau''|a'\tau'\rangle_\Lambda=\int\langle a''\tau''|a\tau\rangle_\Lambda \mathfrak{D}_{\alpha\beta}{}^\tau a\langle a\tau|a'\tau'\rangle_\Lambda\,, \quad \text{(IV.38)}$$

where τ is anyone of the surfaces of the lattice. Let it be the surface nearest to τ''. Then according to (1) as we refine the lattice the various limits exist. If we insert the asymptotic expansion (IV.36) we find in fact the conditions (IV.37).[29]

Connection Between Original and Reduced Amplitudes

Up to here we considered only the reduced amplitude. The connection between the original and the reduced amplitude is given by (IV.9). The reduced amplitude has been defined earlier in this section; in order to give a well-defined meaning to the original amplitude we have to specify how the integrations over Λ in (IV.9) must be carried out. Consider two histories $\Lambda^\mu(x)$ and $\bar\Lambda^\mu(x)$ which reduce to $\Lambda^\mu=\bar\Lambda^\mu$ near τ' and near τ''. The reduced amplitudes associated with these histories differ only in the shape of the lattice in Λ space. Therefore, according to (2) we have

$$\langle a''\tau''|a'\tau'\rangle_\Lambda=\langle a''\tau''|a'\tau'\rangle_{\bar\Lambda}\,, \quad \text{(IV.39)}$$

as was the case for the electromagnetic reduced amplitude. This shows again manifestly that the integrations over the gauge group in the interior of the lattice diverge and we replace them by an average. In the interior of the lattice we may simply drop $\mathfrak{N}_\Lambda\mathfrak{D}\Lambda$ in (IV.9), by virtue of (IV.39). Only in the neighborhood of τ'' are the Λ integrations not trivial and we will investigate these in the next section.

In terms of the original variables of integration, $g_{\mu\nu}(x)$, the measure $\mathfrak{D}\Lambda$ takes the form

$$\mathfrak{D}\Lambda=\prod_L \tfrac{1}{2}|\det g|^{-1}\prod_\mu dg_{\mu 0}\,. \quad \text{(IV.40)}$$

This shows that the average over $\mathfrak{D}\Lambda$ is not equivalent to averaging over $g_{\mu0}$, but includes a weighting factor $|\det g|^{-1}$.

[29] Note that the consistency requirement determines only $\mathfrak{D}_{\alpha\beta}a$. The relation between $\mathfrak{D}_{\alpha\beta}a$ and the original measure $\mathfrak{D}_{\alpha\beta}g$ depends on the particular way one averages over the gauge group. See the discussion following Eq. (IV.40).

Composition Law in Terms of Integration Over Intrinsic Geometries

In particular consider the composition law. From (IV.9) one finds

$$(\mathbf{g}''\tau''|\mathbf{g}'\tau')=\int(\mathbf{g}''\tau''|\mathbf{g}\tau)\mathfrak{D}^\tau\mathbf{g}(\mathbf{g}\tau|\mathbf{g}'\tau')\,,$$
$$\mathfrak{D}^\tau\mathbf{g}=\prod_\tau M|\det\mathbf{g}|^{-1/2}\prod_{i\le k} dg_{ik}\,. \quad \text{(IV.41)}$$

The exponentials carrying the dependence of the amplitudes on $g_{\mu0}$ [Eq. (III.5)] cancel and the average over $g_{\mu0}$ on τ has been carried out with the help of

$$\mathcal{J}[\mathbf{g}]=\int|\det g|^p\prod_\mu dg_{\mu0}=\text{const.}|\det\mathbf{g}|^{p+\frac{1}{2}}. \quad \text{(IV.42)}$$

(This formula may be obtained from the fact that $\mathcal{J}[\mathbf{g}]$ transforms according to

$$\mathcal{J}[\alpha\mathbf{g}\alpha^t]=|\det\alpha|^{2p+1}\mathcal{J}[\mathbf{g}]\,, \quad \text{(IV.43)}$$

if one performs the transformation

$$\mathbf{g}\to\alpha\mathbf{g}\alpha^t \quad g_{i0}\to\alpha_i{}^k g_{k0} \quad g_{00}\to g_{00}.) \quad \text{(IV.44)}$$

The following interesting remark which is due to Wheeler arises in this connection. As has been shown by Baierlein, Sharp, and Wheeler,[30] the specification of the intrinsic geometries of two hypersurfaces together with the requirement that the four-geometry in between satisfy Einstein's equations determines the proper-time separation of the two hypersurfaces. Therefore, if one integrates over all geometries of the surface τ in (IV.41), holding the geometry on τ' fixed he also integrates over all proper-time separations of the two hypersurfaces. Therefore the composition law (IV.41) may be interpreted as an analog of

$$(q''t''|q't')=\text{Const.}\int(q''t''|qt)dqdt(qt|q't')\,.$$

The question whether the composition law for the full gravitational transition amplitude may be brought into a form more closely analogous to the usual relation

$$(q''t''|q't')=\int(q''t''|qt)dq(qt|q't')$$

remains open.

Invariance Under Coordinate Transformations

Finally, let us investigate the behavior of the original amplitude under transformations of the coordinate system. Consider a transformation of coordinates, which reduces to the identity mapping near τ' and τ''. This

[30] R. F. Baierlein, D. H. Sharp, and J. A. Wheeler, Phys. Rev. **126**, 1864 (1962).

transformation is equivalent to a change of the lattice, since we defined the amplitude in terms of the reduced one, where this is obviously the case. By virtue of (2) such a change does not affect the reduced amplitude; therefore we have the result that the original amplitude is invariant with respect to transformations which reduce to the identity mapping near τ' and τ''. Transformations which do not reduce to the identity mapping near τ' and τ'' are responsible for the subsidiary conditions which we shall investigate in the next section. Before we do this let us briefly consider two special cases of such transformations.

First let us investigate a stretching of the lattice in the direction of x^0, τ'' being shifted to τ'''. Furthermore choose the stretching such that only the interior of the lattice is affected, the neighborhood of τ' being left unchanged while the neighborhood of τ'' is shifted rigidly as a whole. By virtue of (2), the reduced amplitude is invariant with respect to such a change of the lattice and we immediately infer that

$$(\mathbf{g}''\tau'''|\mathbf{g}'\tau')=(\mathbf{g}''\tau''|\mathbf{g}'\tau'). \tag{IV.45}$$

In other words the amplitude is independent of τ'' and likewise, of course, of τ'. This property was first emphasized by Misner.

As a second example, consider a transformation

$$\bar{x}^0=x^0;\quad \bar{x}^i=f^i(x^k). \tag{IV.46}$$

It may be verified that the reduced amplitude is invariant with respect to this transformation, if one chooses

$$M=(\Delta^3)^3. \tag{IV.47}$$

If one were to absorb M in N_a then the reduced amplitude would transform like a density rather than be an invariant.[31] Note that there are $N+1$ factors N_a as compared to N factors M in the definition of the reduced amplitude. The necessity of the choice (IV.47) shows up most clearly in the infinitesimal reduced amplitude

$$\langle a''\tau'+\epsilon|a'\tau'\rangle=N_a e^{iS^e}. \tag{IV.48}$$

Clearly S^e is invariant with respect to (IV.46) and so must be N_a; but N_a as given by (IV.37) is only invariant if the factors Δ^3 are absorbed in M according to (IV.47).

We again conclude that the original amplitude is invariant with respect to the transformations (IV.46). This invariance is reflected in the composition law (IV.41). The measure $\mathfrak{D}^\tau\mathbf{g}$ is invariant by virtue of the transformation properties of M.

V. SUBSIDIARY CONDITIONS IN DIFFERENTIAL FORM

Proper Derivation of the Subsidiary Conditions

The way we derived the subsidiary conditions in Sec. III was formal and not satisfactory. Since we now have at our disposal a proper infinitesimal propagator, let us briefly show that we can derive these conditions in the framework of the reduced amplitude in a more satisfactory manner, in the same way as for the electromagnetic amplitude, by analyzing the amplitude for transiting from the surface $\tau''-\epsilon$ to τ''. We have seen that Feynman's definition of the infinitesimal propagator requires the value of the action for the path stationary with respect to both the variations in a_{ik} and in Λ^μ. Since the history Λ^μ appears only in the very last infinitesimal amplitude, by virtue of (IV.39), the condition of stationarity with respect to Λ^μ is trivially satisfied in the interior of the lattice. The average over Λ^μ in the interior thus amounts simply to dropping $\mathfrak{N}_\Lambda\mathfrak{D}\Lambda$ there.

We have to investigate only the variations of the last infinitesimal amplitude with respect to Λ^μ. Moreover, only the values of Λ^μ at τ'' appear in this propagator, such that we may restrict ourselves to a variation $\delta\Lambda^\mu$ which is different from zero say only in the interval $(\tau''-\frac{1}{2}\epsilon,\tau'')$. We have to express the last propagator in the variables $\Lambda^{\mu\prime\prime}$, g_{ik}'' on τ'' and require it to be stationary with respect to this change in Λ^μ for fixed g_{ik}''.

(a) *Change in the Classical Action*

Since we have also to take into account the change in $\chi_{\tau''}(\Lambda'',g'')$ [Eq. (IV.9)], it is easiest to deal directly with the change in the original action $\int\mathfrak{L}(g)d^4x$ [Eq. (IV.7)].

$$\delta\int_{\tau''-\epsilon}^{\tau''}\mathfrak{L}(a)d^4\Lambda+\delta\chi_{\tau''}(\Lambda'',g'')=\delta\int_{\tau''-\epsilon}^{\tau''}\mathfrak{L}(g)d^4x. \tag{V.1}$$

In order to keep g_{ik}'' fixed, we have to vary a_{ik} as well as Λ^μ. The variation in $g_{\mu\nu}(x)$ is given by

$$\begin{aligned}\delta g_{\mu\nu}(x)=(\partial_\mu\delta\Lambda^\alpha\partial_\nu\Lambda^\beta+\partial_\nu\delta\Lambda^\alpha\partial_\mu\Lambda^\beta)a_{\alpha\beta}\\+\partial_\mu\Lambda^i\partial_\nu\Lambda^k(\partial_\lambda a_{ik}\delta\Lambda^\lambda+\delta^*a_{ik}).\end{aligned} \tag{V.2}$$

This may be written

$$\begin{aligned}\delta g_{\mu\nu}(x)=\nabla_\mu\delta\lambda_\nu+\nabla_\nu\delta\lambda_\mu+\partial_\mu\Lambda^i\partial_\nu\Lambda^k\delta^*a_{ik},\\ \delta\lambda^\nu=\Lambda^{-1}{}_\mu{}^\nu\delta\Lambda^\mu.\end{aligned} \tag{V.3}$$

∇_μ denotes the covariant derivative with respect to $g_{\mu\nu}$. To satisfy the requirement $\delta g_{\mu\nu}=0$ on τ'', we have to choose δ^*a_{ik} and the time derivatives of $\delta\Lambda^\mu$ appropriately. We shall not need the explicit solution of this algebraical restriction. Making use of (VI.3) one finds in view of $\delta g_{\mu\nu}=0$ on τ''

$$\begin{aligned}\delta\int_{\tau''-\epsilon}^{\tau''}\mathfrak{L}(g)d^4x&=2\int_{\tau''}S_\mu{}^\nu|g|^{1/2}d\sigma_\nu\delta\lambda^\mu\\&\quad+\int S^{\mu\nu}|g|^{1/2}\partial_\mu\Lambda^i\partial_\nu\Lambda^k\delta^*a_{ik}d^4x\\&=2\int_{\tau''}S_\mu{}^0|g|^{1/2}\delta\lambda^\mu d^3x.\end{aligned} \tag{V.4}$$

[31] These formal considerations are very unsatisfactory: the problem of how to separate the normalization constant from the measure in the Feynman sum over histories formulation deserves a more careful study.

The coefficient of $\delta^* a_{ik}$ vanishes, since a_{ik} satisfies the field equations (IV.11).

(b) *Change in the Measure*

This is the change in the action. What we now have to consider is the change in the measure produced by such a variation. We have seen that the measure is determined by the coefficient of $1/\epsilon$ in the asymptotic expansion of the reduced action in ϵ. We therefore have to look for the change in this leading term. As we remarked earlier, the term $\chi_{\tau''}$ in (IV.7) involves only the boundary values g_{ik}'' and $\Lambda^{\mu''}$ and their partial derivatives with respect to x^1, x^2, x^3. Therefore the variation $\delta\chi_{\tau''}$ will be of order one and give no contribution to the leading term $o(1/\epsilon)$. Thus the relevant part of the variation of the reduced action is given by (V.4). Furthermore, only the term $\delta\lambda^0$ contributes, since S_i^0 contains $\dot{g}_{ik}$ linearly and is therefore $o(\epsilon^{-1/2})$. Thus the leading term in the change of the reduced action is given by

$$\delta\int \mathfrak{L}(a)d^4\Lambda = \frac{1}{4}\int g^{ik}g^{lm}(\dot{g}_{ik}\dot{g}_{lm}-\dot{g}_{il}\dot{g}_{km}) \times |g|^{1/2}d^3x\delta\lambda^0 + o(\epsilon^{-1/2}).$$

Inserting the definition of b_{ik} (V.4) and again neglecting higher order terms, we find

$$\delta\int \mathfrak{L}(a)d^4\Lambda = -\frac{1}{4}\int b^{ik}b^{lm}(b_{il}'b_{km}'-b_{ik}'b_{lm}') \times |b|^{1/2}K\frac{\delta\lambda^0}{\epsilon}d^4x + o(\epsilon^{-1/2}). \quad \text{(V.5)}$$

Therefore the change in the measure will be found by replacing K by $K(1-(\delta\lambda^0/\epsilon))$. Since the factor $K\Delta^3/\epsilon$ appears in the third power in the measure, the change in $\mathfrak{D}a$ is given by

$$\mathfrak{D}a+\delta\mathfrak{D}a = \mathfrak{D}a\prod_L (1-(\delta\lambda^0/\epsilon))^3 = \mathfrak{D}a\left(1-(3/\epsilon\Delta^3)\int\delta\lambda^0 d^3x\right). \quad \text{(V.6)}$$

(c) *Correct Subsidiary Conditions*

With the help of (V.4) and (V.6) we obtain the following modified subsidiary conditions

$$(\mathbf{g}''\tau''|S_i^0|g|^{1/2}(x)|\mathbf{g}'\tau') = 0, \qquad x\epsilon\tau'' \quad \text{(V.7)}$$
$$(\mathbf{g}''\tau''|S_0^0|g|^{1/2}(x)|\mathbf{g}'\tau') = (3i/2\Delta^3\epsilon)(\mathbf{g}''\tau''|\mathbf{g}'\tau').$$

The term on the right-hand side of the fourth subsidiary condition comes from the transformation properties of the measure; since we disregarded changes in the measure completely in our heuristic derivation in Sec. III, such a modification is reasonably to be expected from the present more detailed analysis. As ϵ tends to zero the modification term will not have a well-defined limit. However we will see shortly that the left-hand side of the fourth subsidiary condition will blow up as well and it is a very satisfactory feature of this approach that these divergencies exactly cancel. The divergence in the left-hand side is due to the fact that S_0^0 is quadratic in the velocities $\dot{g}_{ik}$. Already the matrix element of the kinetic energy of a nonrelativistic free particle, $\frac{1}{2}m\dot{\mathbf{x}}^2$ leads to a divergent term. In fact, the procedure given by Feynman to define the matrix element of the kinetic energy unambiguously as the change in the transition amplitude produced by a change in the mass of the particle leads to exactly the same kind of cancellation of the divergent term in the matrix element against the change in the measure. Thus, while the separation of the fourth subsidiary condition into a right- and left-hand side has no well-defined limit, the subsidiary condition as such does have a well-defined limit.

Evaluation of the Matrix Elements Occurring in the Subsidiary Conditions

With the correct matrix form of the subsidiary conditions at hand the next step is to bring them into differential form, as was done in the electromagnetic case.

From unitarity or directly from the fact that we may apply the same arguments if we choose the Gaussian coordinate system to coincide with the lattice system at τ'' instead of τ' we conclude that the subsidiary conditions are valid at τ' as well and we prefer to evaluate them at τ'. The only difference is the sign of the term arising from the transformation properties of the measure. Let us first look at

$$(\mathbf{g}''\tau''|S_i^0(x)|g|^{1/2}|\mathbf{g}'\tau') = 0, \quad x\epsilon\tau', \quad i=1,2,3. \quad \text{(V.8)}$$

Since $\partial_\mu\Lambda^\nu = \delta_\mu^\nu$ at τ' we may as well use the metric $a_{\mu\nu}$ in S_i^0 which then takes the form

$$S_i^0 = \tfrac{1}{2}a^{lm}(D_l\dot{a}_{im} - D_i\dot{a}_{ml}). \quad \text{(V.9)}$$

Here D_l denotes the covariant derivative with respect to the metric a_{ik}. Since $x\epsilon\tau'$ we have $a_{ik}=g_{ik}'$ and all we have to compute is the matrix element of $\dot{a}_{ik}$ for $x\epsilon\tau'$. As a first step we evaluate the reduced matrix element

$$\langle a''\tau''|\dot{a}_{ik}(x)|a'\tau'\rangle_\Lambda, \quad x\epsilon\tau'. \quad \text{(V.10)}$$

(a) *The Matrix Element of* $\dot{a}_{ik}(x)$

The matrix element of $\dot{a}_{ik}$ has the form

$$\lim_{\epsilon\to 0} N_a\int \langle a''\tau''|a\tau'+\epsilon\rangle_\Lambda e^{iS^\epsilon(a,\tau'+\epsilon|a'\tau')_\Lambda}\mathfrak{D}a^{\tau'+\epsilon}\dot{a}_{ik}(\tau').$$

$\Lambda^\mu(x)$ is a given fixed history, which remains the same when we take the limit $\epsilon\to 0$. As a first step we want to show that because we are considering a matrix element at τ', where $\partial_\mu\Lambda^\nu = \delta_\mu^\nu$, we may replace the exponential by its value at $\Lambda^\mu(x)=x^\mu$. The reason is the following: The expansion for $\Lambda^\mu(x)$ reads

$$\Lambda^\mu(\mathbf{x},\tau)=x^\mu+\tfrac{1}{2}\partial_{00}\Lambda^\mu(\mathbf{x},\tau')(\tau-\tau')^2+o(\epsilon^3).$$

If we expand the exponential around the point $\Lambda^\mu(x)=x^\mu$ the first term in this expansion will be linear in $\partial_{00}\Lambda^\mu$ and proportional to ϵ. Since we have subsequently to average over the histories $\Lambda^\mu(x)$, this term will give no contribution by virtue of symmetry. Only the next order which is proportional to ϵ^2 gives nonvanishing contributions. Fortunately we are not interested in this order. Even for the evaluation of the fourth subsidiary condition only the terms of order ϵ contribute. Therefore we may use the asymptotic expansion of the action for the special case $\Lambda^\mu(x)=x^\mu$.

From the way we derived the subsidiary conditions it is clear how we have to evaluate the matrix element of $\dot{a}_{ik}$. We have to compute the slope $\dot{a}_{ik}$ at the surface τ' for the extremal history connecting a_{ik}' on the surface τ' to a_{ik} on $\tau'+\epsilon$. Since we only get contributions if $a_{ik}-a_{ik}'=o(\epsilon^{1/2})$, we may approximate $\dot{a}_{ik}(x)(x\epsilon\tau')$ with the same asymptotic expansion in ϵ. This expansion may be obtained from the Taylor series

$$a_{ik}(\tau'+\epsilon)=a_{ik}(\tau')+\epsilon\dot{a}_{ik}(\tau')+(\tfrac{1}{2}!)\epsilon^2\ddot{a}_{ik}(\tau')+\cdots. \quad \text{(V.11)}$$

Using the equations of motion to express $\ddot{a}_{ik}$ in terms of $\dot{a}_{ik}$ and the notation

$$\Delta a_{ik}=a_{ik}-a_{ik}'=o(\epsilon^{1/2});\quad a_{ik}'=a_{ik}(\tau'), \quad \text{(V.12)}$$

one obtains

$$\begin{aligned}\dot{a}_{ik}(\tau')&=(\Delta a_{ik}/\epsilon)+\delta\dot{a}_{ik},\\ \delta\dot{a}_{ik}&=(1/4\epsilon)a^{lm\prime}(\Delta a_{ik}\Delta a_{lm}-2\Delta a_{il}\Delta a_{km})\\ &\quad-(a_{ik}'/16\epsilon)a^{lm\prime}a^{rs\prime}(\Delta a_{lm}\Delta a_{rs}-\Delta a_{lr}\Delta a_{ms})+o(\epsilon^{1/2}).\end{aligned} \quad \text{(V.13)}$$

Thus we have

$$\langle a''\tau''|\dot{a}_{ik}|a'\tau'\rangle=\lim_{\epsilon\to0}N_a\int\langle a''\tau''|a\tau'+\epsilon\rangle\times e^{iS^e(a\tau'+\epsilon|a'\tau')}((\Delta a_{ik}/\epsilon)+\delta\dot{a}_{ik})\mathfrak{D}a^{\tau'+\epsilon}. \quad \text{(V.14)}$$

This integral may be transformed to the variables $A_\pm{}^l$ and $R_i{}^l$ we introduced in Sec. IV with $a_{ik}^-=a_{ik}$; $a_{ik}^+=a_{ik}'$. The integration over the rotation group is again easy: S^e is independent of $R_i{}^l$ and according to (IV.12) and (IV.19) a_{ik}^+ and a_{ik}^- contribute each a factor $R_i{}^lR_k{}^l$. The integral over such a product of R matrices may be determined from invariance under the rotation group alone, such that we are again left with an integral over the eigenvalues only. This integral is carried out in Appendix 5 with the result

$$\langle a''\tau''|\dot{a}_{ik}(x)|a'\tau'\rangle=-i|a|^{-1/2}(a_{ik}'a_{lm}'-2a_{il}'a_{km}')\times(\delta/\delta a_{lm}'(\mathbf{x}))\langle a''\tau''|a'\tau'\rangle,\quad x\epsilon\tau'.$$

Since the Λ integrations are not affected, this leads to

$$(\mathbf{g}''\tau''|\dot{g}_{ik}(x)|\mathbf{g}'\tau')=-i|g'|^{-1/2}(g_{ik}'g_{lm}'-2g_{il}'g_{km}')\times(\delta/\delta g_{lm}'(\mathbf{x}))(\mathbf{g}''\tau''|\mathbf{g}'\tau'). \quad \text{(V.15)}$$

(b) *Transverse Subsidiary Conditions*

Inserted in (VI.8) one obtains

$$D_l\frac{\delta}{\delta g_{il}'}(\mathbf{g}''\tau''|\mathbf{g}'\tau')=\left(\partial_l\frac{\delta}{\delta g_{il}'}+\begin{Bmatrix} i\\ lm\end{Bmatrix}\frac{\delta}{\delta g_{lm}'}\right)(\mathbf{g}''\tau''|\mathbf{g}'\tau')=0. \quad \text{(V.16)}$$

Note that $\delta/\delta g_{ik}'$ is a contravariant tensor density and transforms like $g^{ik}|g|^{1/2}$ with respect to transformations inside the surface τ'. D_l is the covariant derivative with respect to $g_{ik}'=a_{ik}'$, and $\begin{Bmatrix} i\\ lm\end{Bmatrix}$ the Christoffel symbol formed with the metric g_{ik}' in the space τ'.

Equation (V.16) is the analog of (II.21) in the electromagnetic case. It states that the amplitude $(\mathbf{g}''\tau''|\mathbf{g}'\tau')$ is invariant with respect to an infinitesimal change of the coordinate system on the surface τ'. This simple result confirms the formal considerations at the end of Sec. III. By virtue of unitarity the same must be true on τ''.

Note that in the course of the evaluation of the integral in Appendix 5 we encountered divergencies of the type $1/\Delta^3$, i.e., terms that do not converge to a finite limit when the space-like separation of the lattice, Δ, goes to zero. One such term arises from the correction $\delta\dot{a}_{ik}$ and the others from the asymptotic expansion of the measure and the exponential. These divergencies are related to the fact that we are dealing with an infinite number of degrees of freedom and are a common occurrence in quantum field theory. It is again a very satisfactory result, that they cancel leaving us with an unambiguous invariance condition.

(c) *The Matrix Element of* $S_0{}^0|g|^{1/2}$

To convert the subsidiary condition associated with time translations

$$(\mathbf{g}''\tau''|S_0{}^0(x)|g|^{1/2}|\mathbf{g}'\tau')=-(3i/2\epsilon\Delta^3)(\mathbf{g}''\tau''|\mathbf{g}'\tau'),\quad x\epsilon\tau' \quad \text{(V.17)}$$

into differential form we need

$$S_0{}^0(a)=-\tfrac{1}{2}R^{(3)}+\tfrac{1}{8}a^{ik}a^{lm}(\dot{a}_{ik}\dot{a}_{lm}-\dot{a}_{il}\dot{a}_{km}), \quad \text{(V.18)}$$

where $R^{(3)}$ denotes the curvature scalar formed with the metric a_{ik} and its spatial derivatives $\partial_l a_{ik}$.

The evaluation of (V.17) is more troublesome than the evaluation of (V.8), because $S_0{}^0$ is quadratic in $\dot{a}_{ik}$ and therefore the leading term in the asymptotic expansion is $o(1/\epsilon)$. In order to compute the interesting contributions which are $o(1)$, we have to go up to and including third-order terms in the asymptotic expansion which is an expansion in $\sqrt{\epsilon}$. Second order was sufficient for (V.12) since S_i^0 is linear in $\dot{a}_{ik}$.

To find the asymptotic expansion for S_0^0 we can proceed as follows. If we make use of the equations of motion, $S_i^k(a)=0$, then the Bianchi identities read

$$\partial_0(S_0^0|a|^{1/2})+\partial_i(S_0^i|a|^{1/2})=0. \qquad (V.19)$$

This shows that the change in $S_0^0|a|^{1/2}$ along the extremal path which is given by $\epsilon\partial_0(S_0^0|a|^{1/2})$, is $o(\epsilon^{1/2})$, because S_0^i is linear in $\dot{a}_{ik}$ and therefore $o(\epsilon^{-1/2})$. Thus we may write

$$S_0^0|a|^{1/2}=\frac{1}{\epsilon}\int_{\tau'}^{\tau'+\epsilon} S_0^0|a|^{1/2}d\tau+o(\epsilon^{1/2})$$

$$=-\tfrac{1}{2}R^{(3)}|a|^{1/2}+\frac{1}{8\epsilon}\int a^{ik}a^{lm}(\dot{a}_{ik}\dot{a}_{lm}-\dot{a}_{il}\dot{a}_{km})\times|a|^{1/2}d\tau+o(\epsilon^{1/2}). \qquad (V.20)$$

We have already evaluated this integral in the approximation (IV.10). What remains to be done is to take into account the corrections to the solution a_{ik} arising from the terms $R_{ik}^{(3)}-\frac{1}{2}a_{ik}R^{(3)}$ in (V.11), which we neglected in the above approximation. Let

$$a_{ik}=a_{ik}^0+h_{ik}, \qquad (V.21)$$

where a_{ik}^0 is the approximate solution. Then we have $h_{ik}=o(\epsilon^2)$ since $\ddot{h}_{ik}=o(1)$, $h_{ik}=0$ on τ' and $\tau'+\epsilon$. If we substitute this expression for a_{ik} in the integral in (V.20), consider only terms linear in h_{ik}, and integrate by parts, we obtain only surface terms, because this integral is the Lagrangian for the unperturbed solution a_{ik}^0 which is stationary with respect to a change of a_{ik}^0 in the interior. The surface term vanishes because h_{ik} vanishes at the boundaries τ' and $\tau'+\epsilon$.

This shows that the correction to the integral is quadratic in h_{ik} and therefore negligible. We thus may use the expression (IV.15) for this integral, of course disregarding the integrations on the space variables Λ^1, Λ^2, Λ^3.

(d) *Longitudinal Subsidiary Condition*

The evaluation of the fourth subsidiary condition is now straightforward and carried out in Appendix 6. The result is the following:

$$\mathcal{H}(x)(\mathbf{g}''\tau''|\mathbf{g}'\tau')=0, \quad x\epsilon\tau', \qquad (V.22)$$

$$\mathcal{H}(x)=\mathcal{H}_0(x)+\mathcal{H}_1(x)+\mathcal{H}_2(x), \qquad (V.23)$$

$$\mathcal{H}_0(x)=\tfrac{1}{2}|g'|^{-1/2}(g_{ik}'g_{lm}'-2g_{il}'g_{km}')\times\frac{\delta^2}{\delta g_{ik}'(\mathbf{x})\delta g_{lm}'(\mathbf{x})}+R^{(3)}|g'|^{1/2}(\mathbf{x}), \qquad (V.24)$$

$$\mathcal{H}_1(x)=-\frac{1}{\Delta^3}|g'|^{-1/2}g_{ik}'(\mathbf{x})\frac{\delta}{\delta g_{ik}'(\mathbf{x})}, \qquad (V.25)$$

$$\mathcal{H}_2(x)=-(5/8\Delta^6)|g'|^{-1/2}. \qquad (V.26)$$

The terms in ϵ^{-1} and $\epsilon^{-1/2}$ canceled.[32] Note that the operators $\mathcal{H}_i$ are covariant with respect to transformations in the surface τ', since the volume element of the lattice Δ^3 of course transforms according to $\bar{\Delta}^3=(\partial\bar{x}/\partial x)\Delta^3$, such that $|g'|^{1/2}\Delta^3$ is an invariant. $\mathcal{H}(x)$ is a scalar density of the type $|g'|^{1/2}$. Therefore (V.22) and (V.16) are compatible.

(e) *Remarks on the Singular Terms*

However $\mathcal{H}_1$ and $\mathcal{H}_2$ are divergent as $\Delta\to 0$. They may be interpreted as arising from a particular ordering of the factors in $\mathcal{H}_0$; $1/\Delta^3$ represents $\delta^3(0)$. At first sight one might think they necessarily cause trouble and should be removed by redefining the amplitude. However, one should bear in mind that $\mathcal{H}_0$ is also a singular object. In order to decide whether renormalizations of the amplitude are necessary one has to investigate the solutions of $\mathcal{H}(x)\Psi=0$. Consider, for example, the well-behaved functional

$$\Phi(\varphi)=\exp\left(-\frac{1}{2}\int\varphi^2(\mathbf{x})d^3x\right)=\exp(-\tfrac{1}{2}\sum_L\varphi^2(\mathbf{x}^L)\Delta^3). \qquad (V.27)$$

This functional satisfies the singular-looking differential equation

$$[\delta^2/\delta\varphi^2(\mathbf{x})]\Phi-\varphi^2(\mathbf{x})\Phi=-(1/\Delta^3)\Phi. \qquad (V.28)$$

What is singular with this equation is only the way of separating the differential operator into a right- and a left-hand side. The equation as a whole has well-behaved solutions.

The situation is different in the case of the subsidiary conditions (V.16), because we know the solutions of these conditions. The analog of $\mathcal{H}_0$ is a well-behaved operator there.

The central problem is of course to find the solutions of (VI.22). This problem will not be attacked here. It may be expected that terms proportional to $\mathcal{H}_1$ and $\mathcal{H}_2$ have indeed to be added to $\mathcal{H}_0$ in order that $\mathcal{H}\Psi=0$ has solutions. However the particular coefficients appearing in (V.25) and (V.26) should not be taken too seriously.[33] In particular the coefficient in $\mathcal{H}_2$ is very sensitive to modifications of the definition of the infinitesimal amplitude, since it contains the third-order terms in the expansion of the action and the measure.

[32] The cancellation of these terms is due to the correction term in the fourth subsidiary condition.

[33] Note however that the Hermiticity requirement is only consistent with the particular coefficient of $\mathcal{H}_1$ appearing in (V.25). Compare footnote 25.

VI. CONNECTION TO THE HAMILTONIAN QUANTIZATION PROCEDURE

In this section we want to establish the connection between the sum over histories formulation of the quantum theory of gravity and the canonical quantization procedure, which has been investigated by several authors.[34]

Classical Canonical Theory of Gravity

Let us briefly summarize the relevant aspects of the classical canonical theory of gravity as given by Dirac.[35] The basic elements of this theory are the canonical variables $g_{\mu\nu}(\mathbf{x})$ and $p^{\mu\nu}(\mathbf{x})$, by means of which a Poisson bracket is defined, such that

$$[g_{\mu\nu}(\mathbf{x}),p^{\rho\sigma}(\mathbf{y})]=\delta_{\mu\nu}{}^{\rho\sigma}\delta(\mathbf{x}-\mathbf{y}),\quad \delta_{\mu\nu}{}^{\rho\sigma}=\tfrac{1}{2}(\delta_\mu{}^\rho\delta_\nu{}^\sigma+\delta_\nu{}^\rho\delta_\mu{}^\sigma). \tag{VI.1}$$

Since the Lagrangian associated with the gravitational field is degenerate, the variables $p^{\mu\nu}$ and $g_{\mu\nu}$ are subject to constraints. The Lagrangian $\mathfrak{L}^*$ proposed by Dirac has been chosen in such a way that the algebraic constraints on $p^{\mu\nu}$ read

$$p^{\mu 0}\approx 0. \tag{VI.2}$$

These relations reflect the fact that it is not possible to express $\partial_0 g_{\mu 0}$ in terms of $p^{\mu 0}$.

On the other hand, the restrictions on the initial values $\partial_0 g_{\mu\nu}$ imposed by $S^{\mu 0}=0$ lead to

$$\mathfrak{H}^k\equiv D_i p^{ik}\approx 0, \tag{VI.3}$$

$$\mathfrak{H}_L\equiv -\tfrac{1}{2}|\mathbf{g}|^{-1/2}(g_{ik}g_{lm}-2g_{il}g_{km})p^{ik}p^{lm}+R^{(3)}|\mathbf{g}|^{1/2}\approx 0, \tag{VI.4}$$

where $|\mathbf{g}|$ denotes the determinant of the intrinsic geometry g_{ik} of the surface $x^0=$ constant and D_i is the covariant derivative with respect to this geometry.

Quantized Canonical Theory of Gravity

In the quantized version of the canonical theory the canonically conjugate Hermitian operators $\hat{p}^{\mu\nu}(\mathbf{x})$ and $\hat{g}_{\rho\sigma}(\mathbf{y})$ satisfy

$$[\hat{g}_{\mu\nu}(\mathbf{x}),\hat{p}^{\rho\sigma}(\mathbf{y})]=i\delta_{\mu\nu}{}^{\rho\sigma}\delta(\mathbf{x}-\mathbf{y}), \tag{VI.5}$$

where [,] now denotes the commutator. The constraints (VI.2), (VI.3), and (VI.4) are replaced by restrictions on the state vectors

$$\hat{p}^{\mu 0}\Psi=0, \tag{VI.6}$$

$$\hat{\mathfrak{H}}^i\Psi=0, \tag{VI.7}$$

$$\hat{\mathfrak{H}}_L\Psi=0. \tag{VI.8}$$

There are many representations of the state vectors and operators in different Hilbert spaces. We want to show that the sum over histories formulation of the quantum theory of gravity provides a particular representation of this canonical picture. It is clear that it is going to be a representation where the operator $\hat{g}_{\mu\nu}(x)$ is diagonal.

(a) *State Vectors*

The space of state vectors of this representation is spanned by the functionals $\Psi[\mathbf{g}]$, which are generated by

$$\Psi[\mathbf{g}]=\int(\mathbf{g}\tau|\mathbf{g}_0\tau_0)\mathfrak{D}^{\tau_0}\mathbf{g}_0\Psi_0[\mathbf{g}_0], \tag{VI.9}$$

where Ψ_0 is an arbitrary functional and $\Psi[\mathbf{g}]$ is independent of τ by virtue of (IV.45). Note that only the intrinsic geometries g_{ik} are involved.

(b) *Operators*

The representation of operators is based upon

$$\hat{O}\Psi[\mathbf{g}]=\int(\mathbf{g}\tau|O|\mathbf{g}_0\tau_0)\mathfrak{D}^{\tau_0}\mathbf{g}_0\Psi_0[\mathbf{g}_0], \tag{VI.10}$$

where the matrix element on the right-hand side denotes a sum over histories expression, the histories being weighted by the value of O at the classical path.[36]

Clearly this definition leads to a diagonal representation of the operator $\hat{g}_{\mu\nu}$. Furthermore $\hat{p}^{\mu 0}\Psi=0$, since the value of $p^{\mu 0}$ at the classical path vanishes. (Note $g_{\mu 0}=\delta_\mu{}^0$ at τ.) This shows that (VI.6) is satisfied by our representation. We have already computed the matrix element of $\hat{g}_{ik}$, Eq. (V.15). In order to find the representation of $\hat{p}^{ik}$, we recall that the classical momentum is defined by

$$p^{ik}=\partial\mathfrak{L}/\partial\dot{g}_{ik}=-\tfrac{1}{2}|\mathbf{g}|^{1/2}(g^{ik}g^{lm}-g^{il}g^{km})\dot{g}_{lm}. \tag{VI.11}$$

Since $\hat{g}_{ik}$ does not commute with $\hat{g}_{ik}$, $\hat{p}^{ik}$ is not uniquely defined by (VI.10), but only up to a term proportional to $\delta^3(0)$

$$\hat{p}^{ik}\psi=(-i(\delta/\delta g_{ik})+(iq/\Delta^3)g^{ik})\Psi, \tag{VI.12}$$

where q is real and independent of $\mathbf{x}$. Let us fix q by the requirement that $\hat{p}^{ik}$ be Hermitian in the measure $\mathfrak{D}^\tau\mathbf{g}$. This leads to

$$q=\tfrac{1}{4}. \tag{VI.13}$$

(c) *Subsidiary Conditions*

The commutation relations (VI.5) are satisfied and all that remains to be verified are the constraints (VI.7) and (VI.8). It is easy to see that the constraints (V.16) are equivalent to a particular factor ordering of (VI.7). In terms of the operator $\hat{p}^{ik}$, (V.22) may be

[34] P. A. M. Dirac, Phys. Rev. **114**, 924 (1959); P. G. Bergman, Rev. Mod. Phys. **33**, 510 (1961); R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **116**, 1322 (1959) and following papers; B. S. DeWitt, J. Math. Phys. **2**, 151 (1961); J. Anderson, Phys. Rev. **114**, 1182 (1959).

[35] P. A. M. Dirac, Proc. Roy. Soc. (London) **A246**, 333 (1958).

[36] See R. P. Feynman, Rev. Mod. Phys. **20**, 267 (1948).

written

$$\left\{-\frac{1}{2}\frac{1}{|\mathbf{g}|^{1/4}}g_{ik}(\hat{p}^{ik}\hat{p}^{lm}-\hat{p}^{il}\hat{p}^{km})g_{lm}\frac{1}{|\mathbf{g}|^{1/4}}+R^{(3)}|\mathbf{g}|^{1/2}+\frac{1}{2\Delta^6}|\mathbf{g}|^{-1/2}\right\}\Psi=0. \quad \text{(VI.14)}$$

This displays explicitly the Hermiticity of $\mathcal{H}(x)$ in the measure $\mathfrak{D}^r\mathbf{g}$. The last term in (VI.14) might be included in a different, Hermitian factor ordering of the first term. This shows that our representation satisfies the constraint (VI.8) with a particular ordering of the factors.[37] Note that $\mathcal{H}(x)$ can only be Hermitian in the measure $\mathfrak{D}^r\mathbf{g}$, if $\mathcal{H}_1$ has the numerical coefficient given in (V.25).

VII. SUMMARY

The outstanding feature characterizing the gravitational field is the gauge group of coordinate transformations. The present approach to the quantization of gravity is based on a separation of gauge variables and dynamical variables by means of a transformation to Gaussian coordinates. As a first step a reduced amplitude is constructed; the classical action which characterizes this amplitude is the usual action associated with Einstein's equations, specialized to the particular case of Gaussian variables. Since the gauge group has been removed, this action is nondegenerate and leads to a well-defined reduced amplitude. The second step is to reintroduce the gauge group by summing over all possible transformations to Gaussian coordinate systems. This summation is responsible for the subsidiary conditions. The three transverse subsidiary conditions imply that the physical state vectors associated with the space-like hypersurface τ are functionals of the intrinsic geometry of τ only, independent of the coordinate system chosen to describe this geometry. The longitudinal subsidiary condition states that the Hamiltonian annihilates the physical state vectors. These properties are familiar from Dirac's formulation of the canonical theory of gravity.

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APPENDIX 1: TRANSFORMATION OF THE MEASURE

We want to compute the Jacobian of the transformation of variables $g_{\mu\nu}$ to a_{ik} and $\partial_0\Lambda^\mu$ which is given by

$$g_{ik}=\partial_i\Lambda^l\partial_k\Lambda^m a_{lm}+\partial_i\Lambda^0\partial_k\Lambda^0, \quad (1.1)$$

$$g_{i0}=\partial_i\Lambda^l\partial_0\Lambda^m a_{lm}+\partial_i\Lambda^0\partial_0\Lambda^0, \quad (1.2)$$

$$g_{00}=\partial_0\Lambda^l\partial_0\Lambda^m a_{lm}+\partial_0\Lambda^0\partial_0\Lambda^0. \quad (1.3)$$

[37] The factor ordering problem in the quantum theory of gravity in terms of canonical variables has been investigated by J. Anderson, Phys. Rev. 114, 1182 (1959).

Let us first hold g_{ik} and a_{ik} fixed and compute the Jacobian of the transformation $g_{\mu 0}\to\partial_0\Lambda^\mu$ given by (1.2) and (1.3). Put

$$\prod_\mu dg_{\mu 0}=J_1\prod_\mu d(\partial_0\Lambda^\mu). \quad (1.4)$$

The four-by-four determinant J_1 has the form

$$J_1=\begin{vmatrix}\partial_i\Lambda^l a_{lm} & \partial_i\Lambda^0\\ 2\partial_0\Lambda^l a_{lm} & 2\partial_0\Lambda^0\end{vmatrix} =2\det|\partial_\mu\Lambda^\lambda a_{\lambda\nu}|=2\det\Lambda\det a. \quad (1.5)$$

To find the Jacobian of the transformation $g_{ik}\to a_{ik}$ defined by (1.1) put

$$\prod_{i\leq k} dg_{ik}=J_2\prod_{i\leq k} da_{ik}. \quad (1.6)$$

J_2 depends only on the matrix $\partial_i\Lambda^k=(\tilde{\Lambda})_i{}^k$, since the terms $\partial_i\Lambda^0\partial_k\Lambda^0$ in (1.1) are irrelevant. What we want to show first is that J_2 must be a function of $\det\tilde{\Lambda}$. This result may be obtained with the help of an argument due to Bargmann.[19] Consider the transformation

$$\mathbf{g}'=\tilde{\Lambda}'\mathbf{g}\tilde{\Lambda}'^t. \quad (1.7)$$

We may write the Jacobian of the transformation $g_{ik}'\to a_{ik}$ in two different ways as

$$J_2(\tilde{\Lambda}'\tilde{\Lambda})=J_2(\tilde{\Lambda}')J_2(\tilde{\Lambda}), \quad (1.8)$$

where the left-hand side is the result of carrying out the transformation of the differentials in one step, while the right-hand side is obtained by considering the two successive transformations $g_{ik}'\to g_{ik}$; $g_{ik}\to a_{ik}$. This shows that J_2 is a one-dimensional representation of the linear group in three dimensions and must therefore have the form

$$J_2=|\det\tilde{\Lambda}|^p. \quad (1.9)$$

Considering the special transformation $\tilde{\Lambda}=\lambda 1$ and counting powers of λ one finds $p=4$. Therefore we have

$$\prod_{\mu\leq\nu} dg_{\mu\nu}=2|\det\Lambda||\det\tilde{\Lambda}|^4\det a \times\prod_{i\leq k} da_{ik}\prod_\mu d(\partial_0\Lambda^\mu). \quad (1.10)$$

APPENDIX 2: THE EQUATIONS OF MOTION FOR THE GAUSSIAN METRIC AND THE VALUE OF THE EXTREMAL ACTION BETWEEN INFINITESIMALLY NEARBY SURFACES

We want to solve the equations

$$\ddot{a}_{ik}-a_{ik}a^{lm}\ddot{a}_{lm}+\tfrac{1}{2}\dot{a}_{ik}a^{lm}\dot{a}_{lm}-\dot{a}_{il}a^{lm}\dot{a}_{mk} -\tfrac{1}{4}a_{ik}(a^{lm}\dot{a}_{lm})^2+\tfrac{3}{4}a_{ik}a^{lm}a^{rs}\dot{a}_{lr}\dot{a}_{ms}=0. \quad (2.1)$$

The dot denotes derivative with respect to τ and we are interested in a solution connecting the boundary values a_{ik}^- and a_{ik}^+ at τ^- and τ^+, respectively. Let us introduce the positive definite, unimodular matrix u by

$$a_{ik}=-A^{4/3}u_{ik}, \quad \det u=1. \quad (2.2)$$

The equations of motion for A and u_{ik} are

$$A+\tfrac{3}{32}Au^{ik}u^{lm}u_{il}u_{km}=0\,, \qquad (2.3)$$

$$A\ddot{u}_{ik}-A\dot{u}_{il}u^{lm}\dot{u}_{mk}+2\dot{A}\dot{u}_{ik}=0\,. \qquad (2.4)$$

Equation (2.4) may be integrated once to give

$$\dot{u}_{il}u^{lk}A^2=C_i{}^k\,, \qquad (2.5)$$

where $C_i{}^k$ are the constants of integration. Inserting this result in (2.3) and using the notation

$$C_i{}^kC_k{}^i=C\times C\,, \qquad (2.6)$$

one obtains

$$\ddot{A}+\tfrac{3}{32}C\times C(1/A^3)=0\,. \qquad (2.7)$$

This may be integrated with the result

$$A^2=D(\tau-\tau_0)^2-\tfrac{3}{32}(C\times C/D)\,, \qquad (2.8)$$

D and τ_0 denoting the two constants of integration. To solve (2.5) let us perform a transformation of the independent variable by

$$d\sigma=d\tau A^{-2}\,;$$
$$\sigma=-\int_{\tau}^{\tau^+}[D(t-\tau_0)^2-\tfrac{3}{32}(C\times C/D)]^{-1}dt\,. \qquad (2.9)$$

Then Eq. (2.5) reads

$$du_{ik}/d\sigma=C_i{}^lu_{lk}\,. \qquad (2.10)$$

Since both $u_{ik}(\tau^+)$ and $(du_{ik}/d\sigma)(\tau^+)$ are symmetric and $u_{ik}(\tau^+)$ is positive definite, we may perform a linear transformation which transforms both simultaneously to diagonal form.

$$\bar{u}_{ik}(\tau^+)=\delta_{ik}\quad (d\bar{u}_{ik}/d\sigma)(\tau^+)=u_{(i)}\delta_{ik}\,. \qquad (2.11)$$

This linear transformation leaves the differential equation (2.5) invariant, if we transform C appropriately. $\bar{C}$ is given by

$$\bar{C}_i{}^l=\delta_i{}^lu_{(i)}\,. \qquad (2.12)$$

Thus we obtain

$$d\bar{u}_{ik}(\sigma)/d\sigma=u_{(i)}\bar{u}_{ik}(\sigma)\,, \qquad (2.13)$$

$$\bar{u}_{ik}(\sigma)=e^{u_{(i)}\sigma}\delta_{ik}\,; \qquad (2.14)$$

$\det u=1$ implies

$$\sum_{i=1}^{3}u_{(i)}=C_l{}^l=0\,. \qquad (2.15)$$

On the other hand,

$$C\times C=\sum_{i=1}^{3}u_{(i)}{}^2\,. \qquad (2.16)$$

Therefore, the solution reads

$$u_{ik}(\sigma)=\sum_{l=1}^{3}S_i{}^lS_k{}^le^{u_{(l)}\sigma}\,, \qquad (2.17)$$

where S is the matrix that transforms u_{ik}^- and u_{ik}^+ simultaneously to diagonal form.

Let us compute the action integral with the help of the solutions for A and u.

$$S^e=\int_{ex}\mathfrak{L}(a)d^3xd\tau$$
$$=\frac{1}{4}\int|a|^{1/2}a^{ik}a^{lm}(\dot{a}_{il}\dot{a}_{km}-\dot{a}_{ik}\dot{a}_{lm})d^3xd\tau$$
$$+\int|a|^{1/2}a^{ik}\left(\begin{Bmatrix}l\\ik\end{Bmatrix}\begin{Bmatrix}m\\lm\end{Bmatrix}-\begin{Bmatrix}m\\il\end{Bmatrix}\begin{Bmatrix}l\\mk\end{Bmatrix}\right)d^3xd\tau\,. \qquad (2.18)$$

Here $\begin{Bmatrix}l\\ik\end{Bmatrix}$ denotes the Christoffel symbol associated with the metric a_{ik} and its derivatives ∂_la_{ik}. Note that there are no terms of the type $\dot{a}_{ik}\partial_la_{rs}$ in S^e. Therefore, in the approximation (V.10) we have

$$S^e=\frac{1}{4}\int|a|^{1/2}a^{ik}a^{lm}(\dot{a}_{il}\dot{a}_{km}-\dot{a}_{ik}\dot{a}_{lm})d^3xd\tau+o(\epsilon)\,. \qquad (2.19)$$

Inserting the variables A and u_{ik}, one writes

$$S^e=\frac{1}{4}\int\left(A^2u^{ik}u^{lm}\dot{u}_{il}\dot{u}_{km}-\frac{32}{3}\dot{A}^2\right)d^3xd\tau+o(\epsilon)\,, \qquad (2.20)$$

and making use of the equations of motion for A, one obtains

$$S^e=-\frac{8}{3}\left(\int_{\tau^+}A\dot{A}d^3x-\int_{\tau^-}A\dot{A}d^3x\right)$$
$$=-\frac{8}{3}(\tau^+-\tau^-)\int Dd^3x\,. \qquad (2.21)$$

Thus the only quantity of interest is D which we have to express in terms of the initial and final values a_{ik}^- and a_{ik}^+. Suppose we know the constant $C\times C$ which is related to the initial and final values of u_{ik}. Given $C\times C$ we can express the two constants of integration D and τ_0 in (2.8) in terms of the initial and final values of A. After some algebraic manipulations one obtains

$$D=\epsilon^{-2}[A_+{}^2+A_-{}^2-(4A_+{}^2A_-{}^2+\tfrac{3}{8}C\times C\epsilon^2)^{1/2}]\,;$$
$$\epsilon=\tau^+-\tau^-\,, \qquad (2.22)$$

$$\tau^0=\tfrac{1}{2}(\tau^++\tau^-)-(2\epsilon D)^{-1}(A_+{}^2-A_-{}^2)\,. \qquad (2.23)$$

What remains to be determined is $C\times C=\Sigma u_{(i)}{}^2$. We observed previously that $\exp u_{(i)}(\sigma)$ are the eigenvalues of $u_{ik}(\sigma)$ in the basis where u_{ik}^+ and u_{ik}^- are diagonal. For $\tau=\tau^-$; $\sigma=\sigma^-$ we have in particular that $\exp u_{(i)}(\sigma^-)$

($i=1, 2, 3$) are the eigenvalues of

$$\det|u_{ik}^- - \alpha u_{ik}^+|=0, \quad \alpha_i=\exp u_{(i)}(\sigma^-) \quad (i=1,2,3). \tag{2.24}$$

This equation may be rewritten as

$$\alpha^3-\alpha^2 u^{ik+}u_{ik}^- + \alpha u^{ik-}u_{ik}^+ -1=0. \tag{2.25}$$

The three roots satisfy $\alpha_1\alpha_2\alpha_3=1$ since $\det u=1$. Finally we have to express the quantity $C\times C$ in terms of the three roots of (2.25). As a first step we compute σ^- from the defining Eq. (2.9). The result is

$$\sigma^- = -\int_{\tau^-}^{\tau^+} [D(t-\tau_0)^2 - \tfrac{3}{32}(C\times C/D)]^{-1}dt = -(1/\gamma)\ln J, \tag{2.26}$$

$$\gamma=(\tfrac{3}{8}C\times C)^{1/2}; \quad J=\left|\frac{(4A_-^2A_+^2+\gamma^2)^{1/2}+\gamma}{(4A_+^2A_-^2-\gamma^2)^{1/2}-\gamma}\right|. \tag{2.27}$$

Use has been made of the expressions (2.22) and (2.23) for D and τ_0, and γ replaces the constant $C\times C$. Thus the connection to the eigenvalues α_i is given by

$$\ln\alpha_i = -[u_{(i)}/\gamma]\ln J; \quad \Sigma u_{(i)}^2=(8/3)\gamma^2. \tag{2.28}$$

This relation may be inverted to express J in terms of the eigenvalues

$$\ln J=[\tfrac{3}{8}\sum_{l=1}^{3}(\ln\alpha_l)^2]^{1/2}. \tag{2.29}$$

Since we know the right-hand side from the eigenvalue equation (2.25) we can express γ in terms of known quantities by inverting the relation between J and γ, (2.27), and finally insert the result into the expression for D. In this way one finds for the value of the action at the extremal path

$$S^\epsilon = -\frac{8}{3\epsilon}\int [A_+^2+A_-^2 - A_+A_-(J^{1/2}+J^{-1/2})]d^3x + o(\epsilon), \tag{2.30}$$

where J is given by (2.29).

Let us briefly discuss the question whether the extremal path is unique for given boundary values a_{ik}^+ and a_{ik}^-, i.e., whether the algebraical manipulations we carried out to express the solution in terms of its boundary values might involve equations with more than one root. We already mentioned that the determination of D and τ_0 in terms of A_+ and A_- involved a quadratic equation with two solutions. One of them however leads to a singular solution $A(\tau)$ and is thus excluded. On the other hand the uniqueness of the path $u_{ik}(\tau)$ is guaranteed as long as the three eigenvalues are distinct, because the matrix S is essentially unique if we chose any ordering of these eigenvalues, say $0<\alpha_1<\alpha_2<\alpha_3$. What happens if two of them or all three coincide? This in fact causes no trouble although the matrix S is not unique in this case. We obtain the same path $u_{ik}(\tau)$ for all possible choices of S which diagonalize u_{ik}^+ and u_{ik}^- simultaneously. Different choices of S amount to different linear combinations of identical solutions. Thus, we have the important result that the extremal path is unique in the limit $\epsilon\to 0$.

APPENDIX 3: TRANSFORMATION TO EIGENVALUES IN THE MEASURE

We are interested in the Jacobian of the transformation of the variables of integration a_{ik}^- in the reduced amplitude to eigenvalues A_-^l characterized by

$$a_{ik}^- = -\sum_l S_i^l S_k^l A_-^l, \quad \det S=1, \tag{3.1}$$

$$S=TR; \quad RR^t=1, \quad a^+=-TT^t|\det a_+|^{1/3}. \tag{3.2}$$

Let us first get rid of the fixed matrix T defining

$$a^-=-TCT^t; \quad C_{ik}=\sum_l R_i^l R_k^l A_-^l. \tag{3.3}$$

Using Bargmann's argument as given in Appendix 1 again and noting that $\det T=1$ one obtains

$$\prod_{i\le k} da_{ik} = \prod_{i\le k} dC_{ik}. \tag{3.4}$$

Suppose the matrices R_i^l are expressed in terms of Euler angles as

$$R_i^l=R_i^l(\gamma_1,\gamma_2,\gamma_3). \tag{3.5}$$

We want to compute

$$K=\frac{\partial(C_{11},C_{22},C_{33},C_{23},C_{31},C_{12})}{\partial(A_-^1,A_-^2,A_-^3,\gamma_1,\gamma_2,\gamma_3)}, \tag{3.6}$$

where C_{ik} is given in terms of A_-^l and γ_i by (3.3). Let us use the notation A for the diagonal matrix with the elements A_-^l. Then

$$C=RAR^t, \quad RR^t=1, \tag{3.7}$$

$$dC=Rd\tilde{C}R^t, \tag{3.8}$$

$$d\tilde{C}=dA+[R^tdR,A]. \tag{3.9}$$

Using the same argument again, we have

$$\prod_{i\le k} dC_{ik} = \prod_{i\le k} d\tilde{C}_{ik}. \tag{3.10}$$

Let us look more closely at the commutator terms in $d\tilde{C}$.

$$[R^tdR,A]_{ik}=(R^tdR)_{ik}(A_-^k-A_-^i). \tag{3.11}$$

The diagonal terms vanish. Therefore, we may write

$$d\tilde{C}_{11}d\tilde{C}_{22}d\tilde{C}_{33}=dA_-^1dA_-^2dA_-^3. \tag{3.12}$$

For the off-diagonal elements we find

$$d\tilde{C}_{23}=-(R^tdR)_{23}(A_-^2-A_-^3) \text{ cycl.} \tag{3.13}$$

Thus we have

$$\prod_{i\leq k} da_{ik} = d^3A_- k(\gamma_1,\gamma_2,\gamma_3)d\gamma_1 d\gamma_2 d\gamma_3, \tag{3.14}$$

$$d^3A_- = (A_-^2 - A_-^3)(A_-^3 - A_-^1) \times (A_-^1 - A_-^2)dA_-^1 dA_-^2 dA_-^3. \tag{3.15}$$

To find the function k we may again use invariance arguments. The matrix T as defined in (3.2) is clearly not determined uniquely. If we replace T by

$$T' = TB, \quad BB^t = 1, \tag{3.16}$$

and at the same time put

$$R' = B^t R, \tag{3.17}$$

then nothing will be changed. Since (3.17) amounts to a rotation in the space of the Euler angles we conclude that the measure $k(\gamma_1,\gamma_2,\gamma_3)d\gamma_1 d\gamma_2 d\gamma_3$ has to be the group invariant measure which is unique up to a factor.

$$k(\gamma_1,\gamma_2,\gamma_3)d\gamma_1 d\gamma_2 d\gamma_3 = k_0 d^3R; \quad d^3R = \sin\gamma_2 d\gamma_1 d\gamma_2 d\gamma_3. \tag{3.18}$$

The constant k_0 may be determined, e.g., by means of the following integral

$$I = \int \exp[-\sum_{i,k}(a_{ik})^2] \sum_{i\leq k} da_{ik}, \tag{3.19}$$

which has the value

$$I = (\pi/\sqrt{2})^3. \tag{3.20}$$

On the other hand we can use the transformation (3.1). For convenience let us choose $a_{ik}^+ = -\delta_{ik}$, since a_{ik}^+ is here irrelevant. This amounts to $T=1$, $S=R$. The integral then takes the form

$$I = \int \exp[-\sum_{l=1}^{3}(A_-^l)^2]k_0 d^3A d^3R. \tag{3.21}$$

One has to be somewhat careful with the limits of integration. In order to establish a one-to-one correspondence between a_{ik}^- and its eigenvalues A_-^l and R matrices, let us first of all order the eigenvalues according to $-\infty < A^1 \leq A^2 \leq A^3 < \infty$. With this choice of eigenvalues there are still four different matrices R which transform a given matrix a_{ik}^- to the diagonal form

$$\begin{pmatrix} A^1 & & \\ & A^2 & \\ & & A^3 \end{pmatrix}.$$

Geometrically this means that the main axes of the ellipsoid associated with a given symmetric 3×3 matrix may be chosen in four different ways; the direction of axis number one is fixed only up to a sign and similarly for axis number two. Number three is not independent, since we are considering only rotations with determinant $+1$. If we integrate over the full rotation group, we count each matrix a_{ik}^- four times and have therefore to divide the result by four.

The integral over the rotation group has the value

$$\int_0^{2\pi} d\gamma_1 \int_0^{\pi} d\gamma_2 \int_0^{2\pi} d\gamma_3 \sin\gamma_2 = 8\pi^2, \tag{3.22}$$

and I becomes

$$I = (8\pi^2 k_0/4)\int_{-\infty}^{+\infty} dA_-^1 \int_{A_1}^{\infty} dA_2^- \times \int_{A_2}^{\infty} dA_3^- e^{-\Sigma(A^l)^2}(A_-^2 - A_-^3) \times (A_-^3 - A_-^1)(A_-^1 - A_-^2). \tag{3.23}$$

These integrations can again be carried out with the result

$$I = (\pi/\sqrt{2})^3 k_0. \tag{3.24}$$

Comparison with (3.20) shows that $k_0 = 1$ and the transformation of the measure takes the form

$$\prod_{i\leq k} da_{ik}^- = d^3A_- d^3R. \tag{3.25}$$

APPENDIX 4: THE METHOD OF STATIONARY PHASE

We want to apply the method of stationary phase to the integral (IV.23). Before we do this let us briefly recall the basis of this method in the form which is best suited for our application. Consider an integral of the form

$$J_\epsilon = \int_{-\infty}^{+\infty} e^{i[h(x)/\epsilon]} f(x)dx, \tag{4.1}$$

where $f(x)$ is an infinitely differentiable test function which vanishes faster than any polynomial when $|x|\rightarrow\infty$. $h(x)$ is assumed to be infinitely differentiable and to have one single minimum x_0, such that for any given $\theta>0$ there is a $\delta>0$ such that $|dh/dx|>\delta$ for all x with $|x-x_0|>\theta$. Put

$$f(x) = f_1(x) + f_2(x) \tag{4.2}$$

both infinitely differentiable and $f_1(x)=0$, $|x-x_0|\geq 2\theta$, $f_2(x)=0$, $|x-x_0|\leq\theta$. Then for any p

$$\lim_{\epsilon\to 0}\epsilon^{-p}\left|J_\epsilon - \int_{-\infty}^{+\infty} e^{+i[h(x)/\epsilon]} f_1(x)dx\right| = 0. \tag{4.3}$$

In other words only f_1 is relevant for the asymptotic expansion of J_ϵ in powers of ϵ, a modification of $f(x)$ outside the arbitrarily chosen interval 2θ does not change the asymptotic expansion. [To obtain the result (4.3) take h as a new variable of integration to compute the integral over f_2. This is permissible since the range of integration for f_2 does not include the interval $|x-x_0|\leq\theta$. If one performs $(p+1)$ integrations by parts he finds that the integral over f_2 is indeed $o(\epsilon^{p+1})$.]

A sharper result may be obtained if one assumes $h(x^0)=0$, $(d^2h/dx^2)(x^0)=H\neq 0$. In this case we may perform the change of variable $h(x)=z^2$. The sign of z may be chosen such that the transformation is monotonic. The Jacobian of this transformation never vanishes and at the stationary point $(dz/dx)(x_0)=(H/2)^{1/2}$. J_ϵ takes the form

$$J_\epsilon=\int_{-\infty}^{+\infty}\exp\left(i\frac{z^2}{\epsilon}\right)\varphi(z)dz\,,\quad \varphi(z)=2f|h|^{1/2}\left|\frac{dh}{dx}\right|^{-1/2}. \quad (4.4)$$

The asymptotic expansion of this integral may be obtained by the same kind of estimates as used for (4.3) with the result

$$J_\epsilon=(\pi i\epsilon)^{1/2}\sum_0^\infty\left(\frac{i\epsilon}{4}\right)^n\frac{1}{n!}\left(\frac{d^{2n}\varphi}{dz^{2n}}\right)_{z=0}, \quad (4.5)$$

which is, in general, an asymptotic series. The importance of this result for practical calculations is that we may simply replace $\varphi(z)$ formally by its Taylor series which need not even converge and compute the Fresnel integrals of the powers of z. Aside from the common factor $\sqrt{\epsilon}$ which may be attributed to dz each power of z gives rise to a factor $\sqrt{\epsilon}$.

To apply this calculus to the integral (IV.23), let us first display the variables of integration explicitly by expressing S^e in terms of lattice variables

$$S^e=-\frac{8}{3\epsilon}\int(A_+^2+A_-^2-2A_+A_-\cosh\alpha)d^3x+o(\epsilon)$$
$$=\frac{1}{\epsilon}\sum_L h(A_\pm^{(L)},\alpha_1^{(L)},\alpha_2^{(L)})+o(\epsilon)\,, \quad (4.6)$$

$$h(A_\pm,\alpha_1,\alpha_2)$$
$$=-(8\Delta^3/3)(A_+^2+A_-^2-2A_+A_-\cosh\alpha)\,. \quad (4.7)$$

By virtue of $\alpha_1\alpha_2\alpha_3=1$ only two α's are independent. Expressing (IV.23) in terms of the variables of integration A_-, α_1, and α_2 which are more convenient to discuss the properties of the exponential than A_-^1, A_-^2, and A_-^3, one obtains

$$\Psi'(a^+)=N_a\int\exp\left(i\sum_L\frac{h}{\epsilon}\right)\prod_L M\,A_-^{4(\alpha+\beta+1)+6}$$
$$\times\omega(\alpha_1,\alpha_2)\Psi(A_-,\alpha_1,\alpha_2)dA_-d\alpha_1d\alpha_2\,, \quad (4.8)$$

$$\omega(\alpha_1,\alpha_2)=(\alpha_1\alpha_2)^{-3}(\alpha_2^2\alpha_1-1)(1-\alpha_1^2\alpha_2)(\alpha_1-\alpha_2)\,. \quad (4.9)$$

Let us restrict our attention to one single point of the lattice and consider an integral of the form

$$\Phi_\epsilon=\int\int\int\exp(ih/\epsilon)$$
$$\times\varphi(A_-,\alpha_1,\alpha_2)A_-^\gamma\omega(\alpha_1,\alpha_2)dA_-d\alpha_1d\alpha_2\,. \quad (4.10)$$

h is quadratic in A_- and therefore as a function of A_- for fixed α_1 and α_2 satisfies the necessary requirements to apply (4.3). Thus we obtain only contributions from the point $\partial h/\partial A_-=0$, which is given by

$$A_-^0=A_+\cosh\alpha\,. \quad (4.11)$$

Making use of (4.5) one obtains for the first term in the asymptotic expansion

$$\Phi_\epsilon=(3\pi\epsilon/8i\Delta^3)^{1/2}A_+^\gamma\int\int d\alpha_1d\alpha_2\omega(\alpha_1\alpha_2)$$
$$\times(\cosh\alpha)^\gamma e^{i(g/\epsilon)}\varphi(A_+\cosh\alpha,\alpha_2,\alpha_3) \quad (4.12)$$
$$\times(1+o(\epsilon^{1/2}))\,,$$

$$g(A_+,\alpha_1,\alpha_2)=(8\Delta^3/3)A_+^2(\sinh\alpha)^2\,.$$

In order to discuss the integrations over α_1 and α_2, we recall

$$\alpha=[\tfrac{3}{32}\sum_{l=1}^3(\ln\alpha_l)^2]^{1/2}$$
$$=[\tfrac{3}{16}((\ln\alpha_1)^2+\ln\alpha_1\ln\alpha_2+(\ln\alpha_2)^2)]^{1/2}\,, \quad (4.13)$$

by virtue of $\alpha_1\alpha_2\alpha_3=1$. We need $\partial g/\partial\alpha_1$ in order to verify whether g has the required properties to apply the method of stationary phase. One finds

$$\frac{\partial g}{\partial\alpha_1}=\tfrac{1}{2}A_+^2\Delta^3\frac{\sin\alpha\cos\alpha}{\alpha_1\alpha_2}\ln(\alpha_1^2\alpha_2)\,. \quad (4.14)$$

This expression vanishes only at the point $\alpha_1^2\alpha_2=1$. In view of the restrictions on the range of integration $1/\alpha_1\alpha_2=\alpha_3\geq\alpha_2\geq\alpha_1>0$ this implies $\alpha_2=\alpha_1=1$. Therefore we find only contributions at the totally stationary point $A_-=A_+$, $\alpha_1=\alpha_2=1$. Let us expand φ as well as g and ω around this point. With the notation

$$\ln\alpha_1=-y-z\,,\quad \ln\alpha_2=-y+z\,, \quad (4.15)$$

these expansions are given by

$$g=\tfrac{1}{2}A_+^2\Delta^3(3y^2+z^2)+\text{cubic term}\,, \quad (4.16)$$

$$\omega(\alpha_1\alpha_2)d\alpha_1d\alpha_2$$
$$=4z(9y^2-z^2)dydz+\text{term in fourth order}\,. \quad (4.17)$$

Since higher powers of y and z give rise to higher order terms in the asymptotic expansion in $\sqrt{\epsilon}$ we obtain

$$\Phi_\epsilon=\left[\frac{6\pi\epsilon}{i\Delta^3}\right]^{1/2}A_+^\gamma\varphi(A_+,1,1)\int_0^\infty dy$$
$$\times\int_0^{3y}dz\exp\left[i\frac{\Delta^3}{\epsilon}\frac{A_+^2}{2}(3y^2+z^2)\right]$$
$$\times z(9y^2-z^2)(1+o(\epsilon^{1/2}))\,. \quad (4.18)$$

The range of integration has been adjusted to the restrictions $1/\alpha_1\alpha_2 \geq \alpha_2 \geq \alpha_1 > 0$. The integrations may be carried out and one finds

$$\Phi_\epsilon = 2\pi(\epsilon/\Delta^3)^3 A_+{}^{\gamma-5}\varphi(A_+,1,1). \tag{4.19}$$

Inserted in (4.8) we have the result

$$\Psi(a^+) = N_a \prod_L M\ A_+{}^{4(\alpha+\beta)+5} 2\pi(\epsilon/\Delta^3)^3 \times \bar{\Psi}(A_+,1,1)(1+o(\epsilon^{1/2})). \tag{4.20}$$

APPENDIX 5: THE MATRIX ELEMENT $\langle a''\tau''|\dot{a}_{ik}|a'\tau'\rangle$

There is a slight difference between the integral (IV.16) we considered in Sec. IV and the present one given by (V.14): The integrations now run over the variables a_{ik} which are associated with the upper surface $\tau'+\epsilon$ of the infinitesimal amplitude instead of the lower one as in Sec. IV. Let us again denote the variables which are kept fixed by $a_{ik}{}^+$ and the variables of integration by $a_{ik}{}^-$, i.e., put

$$a_{ik}{}' = a_{ik}{}^+;\quad a_{ik} = a_{ik}{}^-. \tag{5.1}$$

Written in terms of the eigenvalue variables as defined by (IV.12) to (IV.22), the integral (V.14) takes the form

$$\langle a''\tau''|\dot{a}_{ik}|a^+\tau'\rangle = N_a \int \langle a''\tau''|a^-, \tau'+\epsilon\rangle \exp iS^e \times \sum_l S_i{}^l S_k{}^l Q^l \prod_L \frac{M}{4} |\det a^-|^{-1/2} d^3A_- d^3R, \tag{5.2}$$

where

$$\begin{aligned} Q^l &= Q_1{}^l + Q_2{}^l, \\ Q_1{}^l &= -(1/\epsilon)(A_-{}^l - A_+{}^l), \\ Q_2{}^l &= \frac{1}{16\epsilon} A_+{}^{-4/3}[-4(A_-{}^l - A_+{}^l) \times \sum_m (A_-{}^m - A_+{}^m) + 8(A_-{}^l - A_+{}^l)^2 \\ &\quad + \{\sum_m (A_-{}^m - A_+{}^m)\}^2 - \sum_m (A_-{}^m - A_+{}^m)^2], \end{aligned} \tag{5.3}$$

are the eigenvalues of $\dot{a}_{ik}(\tau')$. Note that $A_\pm = |\det a_\pm|^{1/4}$. A factor $\frac{1}{4}$ has again been inserted such that we may now integrate over the full rotation group. We are interested in an asymptotic expansion of (5.2) in ϵ. According to the method of stationary phase the only contributions to this integral arise from the domain $|a_{ik}{}^- - a_{ik}{}^+| = o(\epsilon^{1/2})$. The contributions from the rest of the range of integration are annihilated by destructive interference of the rapidly oscillating exponential. Let us again expand all quantities in a Taylor series around the stationary point. How far do we have to go in this expansion in order to get all terms that do not vanish in the limit $\epsilon \to 0$? The firstt erm in Q^l is $o(\epsilon^{1/2})$ and the remaining ones $0(1)$ since $(A_-{}^l - A_+{}^l) = o(\epsilon^{1/2})$. Therefore, the leading term in the expansion will be $o(\epsilon^{-1/2})$. Second-order terms are ϵ-independent and third or higher orders vanish in the limit $\epsilon \to 0$. Thus we need the expansions to second order. Consider first the amplitude

$$\langle a''\tau''|a^-, \tau'+\epsilon\rangle = \left[1 + \int d^3u (a_{ik}{}^- - a_{ik}{}^+)(\mathbf{u}) \frac{\delta}{\delta a_{ik}{}^+(\mathbf{u})}\right] \times \langle a''\tau''|a^+, \tau'+\epsilon\rangle + o(\epsilon). \tag{5.4}$$

To obtain the expansion for S^e we have to solve the equations of motion to sufficient accuracy. This problem was discussed in Sec. V, where we obtained the result that the action

$$\begin{aligned} S^e &= \int B d^4x + \frac{1}{4}\int |a|^{1/2} a^{ik} a^{lm} (\dot{a}_{ik} a_{lm} - \dot{a}_{il}\dot{a}_{km}) d^4x, \\ B &= |a|^{1/2} a^{ik} \left(\begin{Bmatrix} l \\ ik \end{Bmatrix} \begin{Bmatrix} m \\ lm \end{Bmatrix} - \begin{Bmatrix} l \\ im \end{Bmatrix} \begin{Bmatrix} m \\ lk \end{Bmatrix} \right), \end{aligned} \tag{5.5}$$

may be computed with the solution of the equations of motion in the approximation (IV.10). The corrections to S^e which arise when the equations of motion are solved to higher order of accuracy are $o(\epsilon^3)$ and therefore clearly irrelevant for our purpose, since we need S^e only up to and including $o(\epsilon^{1/2})$. This approximation is still sufficient for the evaluation of the fourth subsidiary condition which we shall attack in Appendix 6, where we need the action up to and including $o(\epsilon)$. To this order the action is given by

$$S^e = \epsilon \int B(a^+) |a^+|^{1/2} d^3x - \tfrac{3}{8}\epsilon^{-1} \times \int (A_+{}^2 + A_-{}^2 - 2A_+A_- \cosh\alpha) d^3x + o(\epsilon^{3/2}). \tag{5.6}$$

To carry out the integration it is convenient to again make use of the variables y and z introduced in Appendix 4. Furthermore we replace the variable A_- by x. The transformation from $A_-{}^l$ to the new set of variables x, y, z is given by

$$A_-{}^1 = A_+{}^{4/3}\exp(\tfrac{4}{3}x - y - z);\quad A_-{}^2 = A_+{}^{4/3}\exp(\tfrac{4}{3}x - y + z); \quad A_-{}^3 = A_+{}^{4/3}\exp(\tfrac{4}{3}x + 2y). \tag{5.7}$$

The action S^e to lowest order is quadratic in x, y, z and $x = y = z = 0$ is the stationary point. Therefore, x, y, and z are $o(\epsilon^{1/2})$. In terms of these variables we have the following expansion for the action S^ϵ

$$S^\epsilon=\int d^3x[S_1(\mathbf{x})+S_2(\mathbf{x})+S_3(\mathbf{x})+\epsilon B(\mathbf{x})]+o(\epsilon^{3/2})\,, \quad (5.8)$$

$$S_1=-\frac{8A_+{}^2}{3\epsilon}x^2+\frac{A_+{}^2}{2\epsilon}(3y^2+z^2)=0(1)\,,$$

$$S_2=xS_1=o(\epsilon^{1/2})\,, \quad (5.9)$$

$$S_3=-\frac{2A_+{}^2}{3\epsilon}\left(\frac{7x^4}{3}-\tfrac{3}{8}(3y^2+z^2)x^2-\frac{3}{256}(3y^2+z^2)^2\right)=o(\epsilon)\,,$$

where we have included third-order terms which we shall need in Appendix 6. For the present purposes we can drop both S_3 and B.

Finally, we have to express the measure in terms of the new variables of integration x, y, z and to expand the nonlinear factors appearing in the measure. The result reads, again correct to third order

$$\prod_L \frac{M}{4}|\det a_-|^{-1/2}d^3A_-d^3R$$
$$=(1+m_1+m_2)\mathfrak{D}x\mathfrak{D}y\mathfrak{D}z\prod_L d^3R(1+o(\epsilon^{3/2}))\,, \quad (5.10)$$

$$\mathfrak{D}x\mathfrak{D}y\mathfrak{D}z=\prod_L 4Mz(9y^2-z^2)dxdydz\,,$$

$$m_1=\sum_L 6x\,, \quad (5.11)$$

$$m_2=\sum_L (3y^2+z^2)/4+\tfrac{1}{2}m_1{}^2.$$

Collecting these expansions and retaining only second-order terms the matrix element of $\dot{a}_{ik}$ becomes

$$\langle a''\tau''|\dot{a}_{ik}|a'\tau'\rangle=M_{ik}{}^1+M_{ik}{}^2+M_{ik}{}^3+o(\epsilon^{1/2})\,, \quad (5.12)$$

$$M_{ik}{}^1=\langle a''\tau''|a^+,\tau'+\epsilon\rangle\langle\sum_l S_i{}^lS_k{}^lQ_1{}^l\rangle\,,$$

$$M_{ik}{}^2=\langle a''\tau''|a^+,\tau'+\epsilon\rangle\left\langle\left(i\int S_2d^3y+m_1\right)\times\sum_l S_i{}^lS_k{}^lQ_1{}^l+\sum_l S_i{}^lS_k{}^lQ_2{}^l\right\rangle\,, \quad (5.13)$$

$$M_{ik}{}^3=\int d^3u\frac{\delta}{\delta a_{lm}{}^+(\mathbf{u})}\langle a''\tau''|a^+,\tau'+\epsilon\rangle\times\langle(a_{lm}{}^- - a_{lm}{}^+)_\mathbf{u}\sum_l S_i{}^lS_k{}^lQ_1{}^l\rangle\,,$$

where we used the notation

$$\langle\varphi\rangle=\int e^{i\int S_1d^3x}\varphi\mathfrak{D}x\mathfrak{D}y\mathfrak{D}z\prod_L d^3R. \quad (5.14)$$

The matrix element $M_{ik}{}^1$ is $o(\epsilon^{-1/2})$ while $M_{ik}{}^2$ and $M_{ik}{}^3$ consist of ϵ-independent terms only. The range of integration includes the full rotation group, the integration over x runs from $-\infty$ to $+\infty$ while the limits in y and z are the same as in Appendix 4.

Let us examine $M_{ik}{}^1$ first. The matrix $R_i{}^k$ appears only in $S_i{}^l$. Since $S=TR$ the integration over the rotation group involves an integral of the form

$$r_{ik}{}^{lm}=\int R_i{}^lR_k{}^md^3R\,, \quad (5.15)$$

which is recognized as a special case of the integral occurring in the group orthogonality relation. If we keep l and m fixed, replace R by R_1R_2 and integrate over R_2 instead of R we obtain, in view of the invariance of the measure d^3R

$$r_{ik}{}^{lm}=R_i{}^rR_k{}^sr_{rs}{}^{lm}\,, \quad (5.16)$$

i.e., $r_{ik}{}^{lm}$ is for fixed upper indices an invariant tensor under the rotation group in the lower ones. This implies

$$r_{ik}{}^{lm}=r^{lm}\delta_{ik}.$$

On the other hand, integrating over R_1 and holding R_2 fixed, the same statements apply to the indices l and m. Therefore,

$$r_{ik}{}^{lm}=r\delta^{lm}\delta_{ik}. \quad (5.17)$$

To determine r let us sum over $l=m$ in (5.15). The integral on the right-hand side reduces to

$$\delta_{ik}\int d^3R=\delta_{ik}8\pi^2\,,$$

because $RR^t=1$. Thus,

$$r=8\pi^2/3. \quad (5.18)$$

Inserting this result in (5.12) we obtain

$$M_{ik}{}^1=(TT^t)_{ik}\langle a''\tau''|a^+,\tau'+\epsilon\rangle\tfrac{1}{3}\langle\textstyle\sum_l Q_1{}^l\rangle\,, \quad (5.19)$$

where we have rewritten $8\pi^2$ as $\int d^3R$ such that the product $\prod_L d^3R$ again runs over all points of the lattice. Expressed in terms of the variables x, y, and z, $\sum_l Q_1{}^l$ becomes

$$\textstyle\sum_l Q_1{}^l=-\epsilon^{-1}A_+{}^{4/3}(4x+(8x^2/3)+3y^2+z^2). \quad (5.20)$$

This shows that $M_{ik}{}^1$ contains also contributions which are independent of ϵ besides the $\epsilon^{-1/2}$ term, which is linear in x. The integral over this term vanishes by symmetry since both S_1 and $\mathfrak{D}x\mathfrak{D}y\mathfrak{D}z$ are even in x. Therefore, the matrix element of a_{ik} does not lead to terms that diverge as $\epsilon\to 0$. For the same reason the terms $\int S_2d^3y=\sum_L S_2(\mathbf{y})\Delta^3$ and m_1 in $M_{ik}{}^2$ do not contribute unless the point $\mathbf{x}$ at which we are evaluating a_{ik} coincides with $\mathbf{y}$. Therefore, we are left with a single term from both these sums. In $M_{ik}{}^2$ we need, besides $\sum Q_1{}^l$ also the quantity $\sum Q_2{}^l$ which arises when the integrations over the rotation group are carried out in M_{ik}^2. Inserting the variables x, y, and z, $\sum Q_2^l$ becomes

$$\sum_l Q_2^l=(A_+^{4/3}/\epsilon)(\tfrac{2}{3}x^2+\tfrac{5}{8}(3y^2+z^2)). \quad (5.21)$$

If one makes use of $TT^tA_+^{4/3}=-a^+$ the sum of the matrix elements M_{ik}^1 and M_{ik}^2 takes the form

$$M_{ik}^1+M_{ik}^2=\tfrac{1}{3}a_{ik}^+\langle a''\tau''|a^+,\tau'+\epsilon\rangle\Bigg\langle\frac{26x^2}{\epsilon}+\frac{3}{8\epsilon}(3y^2+z^2)$$
$$+\frac{4i\Delta^3x^2}{\epsilon}A_+^2\bigg(-\frac{8x^2}{3\epsilon}+\frac{3y^2+z^2}{2\epsilon}\bigg)\Bigg\rangle.$$

The integrations over x, y, and z are now straightforward although somewhat tedious. The result is very simple

$$M_{ik}^1+M_{ik}^2=0,$$

and we are left with M_{ik}^3. Note that the terms in $M_{ik}^1+M_{ik}^2$ are proportional to $1/\Delta^3$ and diverge as $\Delta\rightarrow 0$. The cancellation would not occur in general for a different form of the measure, since the term m_1 is essential in this cancellation.

Finally consider M_{ik}^3. Again we do not get any contribution for $\mathbf{x}\neq\mathbf{u}$, since then we are lead to an integral which is odd in x. Therefore we may drop the integral over $\mathbf{u}$ replacing d^3u by Δ^3.

The integrations over the rotation group involve an integral of the type

$$r_{iklm}{}^{abcd}=\int R_i{}^aR_k{}^bR_l{}^cR_m{}^dd^3R. \quad (5.22)$$

The same argument as applied to $r_{ik}{}^{lm}$ shows that for fixed a, b, c, and d, the quantity $r_{iklm}{}^{abcd}$ must be an invariant tensor and therefore of the form

$$r_{iklm}{}^{abcd}=r_1{}^{abcd}\delta_{ik}\delta_{lm}+r_2{}^{abcd}\delta_{il}\delta_{km}+r_3{}^{abcd}\delta_{im}\delta_{kl}. \quad (5.23)$$

Again interchanging the role of R_1 and R_2 one finds that the coefficients $r_A{}^{abcd}$ must be linear combinations of the direct product of the metric with itself

$$r_A{}^{abcd}=r_{A1}\delta^{ab}\delta^{cd}+r_{A2}\delta^{ac}\delta^{bd}+r_{A3}\delta^{ad}\delta^{bc}. \quad (5.24)$$

The coefficients r_{AB} may be determined by means of contractions. We need only the particular elements (a and b not summed)

$$r_{iklm}{}^{aabb}=\frac{4\pi^2}{15}[(4-2\delta^{ab})\delta_{ik}\delta_{lm}-(1-3\delta^{ab})(\delta_{il}\delta_{km}+\delta_{im}\delta_{kl})]. \quad (5.25)$$

Inserting this result in M_{ik}^3 we find

$$M_{ik}^3=\frac{\delta}{\delta a_{lm}^+(\mathbf{x})}\langle a''\tau''|a^+,\tau'+\epsilon\rangle\times(a_{ik}^+a_{lm}^+M_1+a_{il}^+a_{km}^+M_2), \quad (5.26)$$

$$M_1=(\Delta^3\epsilon/15)A_+^{-8/3}\langle 2(\sum_m Q_1^m)^2-\sum_m(Q_1^m)^2\rangle, \quad (5.27)$$

$$M_2=(\Delta^3\epsilon/15)A_+^{-8/3}\langle-(\sum_m Q_1^m)^2+3\sum_m(Q_1^m)^2\rangle. \quad (5.28)$$

The quantities $\sum_m Q_1^m$ and $\sum_m(Q_1^m)^2$ may again be expressed in terms of the variables x, y and z and the integrations are of the same type as those encountered in the evaluation of M_{ik}^1 and M_{ik}^2. The result is

$$M_1=-i|a'|^{-1/2};\quad M_2=2i|a'|^{-1/2}, \quad (5.29)$$

where use has been made of the value (V.39) for the normalization constant N_a.

Finally, let us go to the limit $\epsilon\rightarrow 0$ in (5.26). Since everything except the amplitude is ϵ-independent we obtain

$$\langle a''\tau''|\dot{a}_{ik}(\mathbf{x})|a'\tau'\rangle=-i|a'|^{-1/2}(a_{ik}'a_{lm}'-2a_{il}'a_{km}')\times\frac{\delta}{\delta a_{lm}'(\mathbf{x})}\langle a''\tau''|a'\tau'\rangle. \quad (5.30)$$

APPENDIX 6: THE MATRIX ELEMENT $\langle a''\tau''|S_0^0|a|^{1/2}|a'\tau'\rangle$

The procedure as given in some detail in Appendix 5 may be applied to the fourth subsidiary condition in a straightforward manner. We merely have to take into account terms up to third order instead of second.

Let us introduce the quantity $\theta(\mathbf{x})$ by

$$-\tfrac{1}{2}\theta=(S_0^0+\tfrac{1}{2}R^{(3)})|a|^{1/2}+(3i/2\Delta^3\epsilon). \quad (6.1)$$

In Sec. V an expression for $S_0^0|a|^{1/2}$ was obtained [Eqs. (V.20) and (V.15)]. Making use of the asymptotic expansion (5.8) we find

$$\theta=\epsilon^{-1}(\bar{S}_1+S_2+S_3);\quad \bar{S}_1=S_1-(3i/\Delta^3). \quad (6.2)$$

The matrix element of $\theta(\mathbf{x})$ becomes

$$\langle a''\tau''|\theta(\mathbf{x})|a'\tau'\rangle=\bigg[\theta_1(\mathbf{x})+\int d^3u\theta_{2ik}(\mathbf{x},\mathbf{u})\frac{\delta}{\delta a_{ik}^+(\mathbf{u})}+\int\int d^3ud^3v\theta_{3iklm}(\mathbf{x},\mathbf{u},\mathbf{v})\frac{\delta^2}{\delta a_{ik}(\mathbf{u})\delta a_{lm}(\mathbf{v})}\bigg]\times\langle a''\tau''|a^+,\tau'+\epsilon\rangle+o(\epsilon^{1/2}), \quad (6.3)$$

$$\theta_1(\mathbf{x})=\epsilon^{-1}\bigg\langle(\bar{S}_1+S_2+S_3)_{\mathbf{x}}\times\bigg(1+i\int S_2d^3z+i\int S_3d^3z+i\epsilon\int Bd^3z-\frac{1}{2}\Big(\int S_2d^3z\Big)^2\bigg)(1+m_1+m_2)\bigg\rangle, \quad (6.4)$$

$$\theta_{2ik}(\mathbf{x},\mathbf{u})=\epsilon^{-1}\Big\langle (a_{ik}^{-}-a_{ik}^{+})_{\mathbf{u}}(\bar{S}_1+S_2)_{\mathbf{x}} \times\Big(1+i\int S_2 d^3z\Big)(1+m_1)\Big\rangle, \quad (6.5)$$

$$\theta_{3iklm}(\mathbf{x},\mathbf{u},\mathbf{v})=\epsilon^{-1}\langle (a_{ik}^{-}-a_{ik}^{+})_{\mathbf{u}} \times(a_{lm}^{-}-a_{lm}^{+})_{\mathbf{v}}(\bar{S}_1)_{\mathbf{x}}\rangle. \quad (6.6)$$

The leading terms in θ_1, θ_2, θ_3 are respectively $o(\epsilon^{-1})$, $o(\epsilon^{-1/2})$, $o(1)$. Consider θ_1 first and write it as

$$\theta_1(\mathbf{x})=\langle \epsilon^{-1}a(\mathbf{x})+\epsilon^{-1/2}b(\mathbf{x})+c(\mathbf{x})\rangle, \quad (6.7)$$

with a, b, and c ϵ-independent

$$a(\mathbf{x})=\bar{S}_1(\mathbf{x}), \quad (6.8)$$

$$b(\mathbf{x})=\epsilon^{-1/2}\Big[\bar{S}_1(\mathbf{x})\Big(\int S_2 d^3z+m_1\Big)+S_2(\mathbf{x})\Big], \quad (6.9)$$

$$c(\mathbf{x})=\epsilon^{-1}\Big[\bar{S}_1(\mathbf{x})\Big\{\int S_3 d^3z+i\epsilon\int B d^3z -\frac{1}{2}\Big(\int S_2 d^3z\Big)^2+i\int S_2 d^3z\, m_1+m_2\Big\} +S_2(\mathbf{x})\Big(i\int S_2 d^3z+m_1\Big)+S_3(\mathbf{x})\Big]. \quad (6.10)$$

The integrations over the rotation group are trivial in θ_1 since the integrand is independent of R. Let us first look at the matrix element of a. Note that $\bar{S}_1$ contains the correction term $3i/\Delta^3$ which accounts for the transformation properties of the measure in the fourth subsidiary condition. The integrations over x, y, and z may be carried out with the result

$$\langle a(\mathbf{x})\rangle=0. \quad (6.11)$$

In other words, θ_1 does not contain a term proportional to $1/\epsilon$. This result would clearly not be true had we not taken the correction term to the fourth subsidiary condition into account. Next consider the matrix element of b. This matrix element vanishes

$$\langle b(\mathbf{x})\rangle=0, \quad (6.12)$$

by virtue of symmetry: S_2 and m_1 are odd in the variable x. Therefore, θ_1 is independent of ϵ. Evaluating the matrix element of c we obtain

$$\theta_1=-\frac{5}{8\Delta^6}|a'|^{-1/2}. \quad (6.13)$$

To compute $\theta_2(\mathbf{x},\mathbf{u})$ we may use the result (5.17), (5.18) for the integrations over the rotation group. Using (6.11) and the same symmetry arguments as above one concludes that only the contributions from $\mathbf{x}=\mathbf{u}$ survive. The result reads

$$\theta_{2ik}(\mathbf{x},\mathbf{u})=-\delta(\mathbf{x}-\mathbf{u})\frac{1}{\Delta^3}|a'|^{-1/2}a_{ik}'(\mathbf{x}) \quad (6.14)$$

and is again ϵ-independent, because the leading term in θ_2 which is proportional to $\epsilon^{-1/2}$, vanishes by symmetry. Note that in our notation $\Delta^{-3}=\delta(0)$.

Finally, $\theta_3(\mathbf{x},\mathbf{u},\mathbf{v})$ may be determined using the expression (5.25) for the integral over four R matrices. The integrations over the variables x, y, z are again straightforward and lead to

$$\theta_{3iklm}(\mathbf{x},\mathbf{u},\mathbf{v}) =\delta(\mathbf{x}-\mathbf{u})\delta(\mathbf{x}-\mathbf{v})\tfrac{1}{2}|a'|^{-1/2} \times(a_{ik}'a_{lm}'-a_{il}'a_{km}'-a_{im}'a_{kl}')(\mathbf{x}). \quad (6.15)$$

If we collect these results, go to the limit $\epsilon\to 0$ and insert the reduced matrix element of $\theta(\mathbf{x})$ in the subsidiary condition (V.17) we are lead to the expressions (V.22) to (V.26), since the Λ integrations are again not affected.